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ISOMORPHISMS OF $\mathcal{A}^{(3)}_{\infty(i,k)}$

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1. Introduction

The study of non-self-adjoint operator algebras on Hilbert space was only beginned by W.B. Arveson[1] in 1974. Recently, such algebras have been found to be of use in physics, in electrical engineering, and in general systems theory. Of particular interest to mathematicians are reflexive algebras with commutative lattices of invariant subspaces. One of the most important classes of such algebras is the sequence of "tridiagonal" algebras, discovered by F. Gilfeather and D. Larson[8]. These algebras possess many surprising properties related to isomorphisms and cohomology, and are not yet well understood.

Let $\mathcal{H}$ be a complex Hilbert space with an orthonormal basis $\{e_1, e_2, e_3, \cdots\}$. Let $\mathcal{A}^{(3)}_{\infty}$ be the algebra consisting of all bounded operators acting on $\mathcal{H}$ of the form:

$$\begin{pmatrix} * & * & & & \\ & * & & & \\ & * & * & * & \\ & & & * & \\ & & & * & \\ & & & & \ddots \end{pmatrix},$$

where all non-starred entries are zero.

Let $\mathcal{A}^{(3)}_{\infty(i,k)}$ be a subalgebra of $\mathcal{B}(\mathcal{H})$ such that an operator A is in $\mathcal{A}^{(3)}_{\infty(i,k)}$ if and only if it is in $\mathcal{A}^{(3)}_{\infty}$ and its two off-diagonal entries

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are zero. First we will introduce the terminologies used in this paper. Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{A}$ be a subset of $\mathcal{B}(\mathcal{H})$, the class of all bounded operators acting on $\mathcal{H}$. If $\mathcal{A}$ is a vector space over C and if $\mathcal{A}$ is closed under the composition of maps, then $\mathcal{A}$ is called an algebra. $\mathcal{A}$ is called a self-adjoint algebra provided A^* is in $\mathcal{A}$ for every A in $\mathcal{A}$. Otherwise, $\mathcal{A}$ is called a non-self-adjoint algebra. If $\mathcal{L}$ is a lattice of orthogonal projections acting on $\mathcal{H}$, $Alg\mathcal{L}$ denotes the algebra of all bounded operators acting on $\mathcal{H}$ that leave invariant every orthogonal projection in $\mathcal{L}$. A subspace lattice $\mathcal{L}$ is a strongly closed lattice of orthogonal projections acting on $\mathcal{H}$, containing 0 and 1. Dually, if $\mathcal{A}$ is a subalgebra of $\mathcal{B}(\mathcal{H})$, then $Lat\mathcal{A}$ is the lattice of all orthogonal projections invariant for each operator in $\mathcal{A}$. An algebra $\mathcal{A}$ is reflexive if $\mathcal{A} = AlgLat\mathcal{A}$ and a lattice $\mathcal{L}$ is reflexive if $\mathcal{L} = LatAlg\mathcal{L}$. A lattice $\mathcal{L}$ is a commutative subspace lattice, or CSL, if each pair of projections in $\mathcal{L}$ commutes; $Alg\mathcal{L}$ is then called a CSL-algebra. If $x_1, x_2, \cdots, x_n$ are vectors in some Hilbert space, then $[x_1, x_2, \cdots, x_n]$ means the closed subspace generated by the vectors $x_1, x_2, \cdots, x_n$.

2. Isomorphisms of $\mathcal{A}^{(3)}_{\infty(i,k)}$

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be commutative subspace lattices. By an isomorphism

$$\varphi : Alg\mathcal{L}_1 \longrightarrow Alg\mathcal{L}_2$$

we mean a strictly algebraic isomorphism, that is a bijective, linear, multiplicative map. An isomorphism $\varphi : Alg\mathcal{L}_1 \longrightarrow Alg\mathcal{L}_2$ is said to be spatially implemented if there is a bounded invertible operator T such that $\varphi(A) = TAT^{-1}$ for all A in $Alg\mathcal{L}_1$ An isomorphism $\varphi : Alg\mathcal{L}_1 \longrightarrow Alg\mathcal{L}_2$ is said to be quasi-spatially implemented if there exists a one-to-one operator T with a dense domain $\mathcal{D}$, that is an invariant linear manifold for $Alg\mathcal{L}_1$, such that $\varphi(A)Tf = TAf$ for all A in $Alg\mathcal{L}_1$ and $f \in \mathcal{D}$. Let i and j be natural numbers. Then E_{ij} is the matrix whose (i,j)-component is 1 and all other components are zero. If $\varphi : \mathcal{A}^{(3)}_\infty \longrightarrow \mathcal{A}^{(3)}_\infty$ is an isomorphism, then we know that φ is quasi-spatially implemented [5]. Let $\mathcal{A}^{(3)}_{\infty(i,k)}$ be a subalgebra of $\mathcal{B}(\mathcal{H})$ such that an operator A is in $\mathcal{A}^{(3)}_{\infty(i,k)}$ if and only if A is in $\mathcal{A}^{(3)}_\infty$

and its $(2i+1,2i)$-, $(2k+1,2k)$- component are zero, where $i<k$, $i=1,2,\cdots$. Let $\mathcal{L}$ be the commutative subspace lattice generated by $\{[e_{2p-1}], [e_1,e_2,e_3], [e_3,e_4,e_5], \cdots, [e_{2i-1},e_{2i}], [e_{2i+1},e_{2i+2},e_{2i+3}], \cdots, [e_{2k-1},e_{2k}], [e_{2k+1},e_{2k+2},e_{2k+3}], \cdots : p=1,2,\cdots\}$. Then $\mathcal{A}^{(3)}_{\infty(i,k)} = Alg\mathcal{L}$. Since $\mathcal{L}$ is commutative, $\mathcal{L}$ and $\mathcal{A}^{(3)}_{\infty(i,k)}$ are reflexive.

THEOREM 1. (Gilfeather and Moore [9]) *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be commutative subspace lattices on Hilbert spaces H_1 and H_2, respectively, and let $\varphi : Alg\mathcal{L}_1 \longrightarrow Alg\mathcal{L}_2$ be an algebraic isomorphism. Then φ is uniformly continuous.*

THEOREM 2. *Let $\varphi : \mathcal{A}^{(3)}_{\infty(i,k)} \longrightarrow \mathcal{A}^{(3)}_{\infty(i,k)}$ be an isomorphism such that $\varphi(E_{pp}) = E_{pp}$ for all $p=1,2,\cdots$. Then there exist nonzero complex numbers $\alpha_{m,n}$ such that $\varphi(E_{m,n}) = \alpha_{m,n}E_{m,n}$ for all $E_{m,n}$ in $\mathcal{A}^{(3)}_{\infty(i,k)}$.*

Proof. Since $\varphi(E_{pp}) = E_{pp}$ and φ is an isomorphism, we have $\varphi(E_{pp}^{\perp}) = E_{pp}^{\perp}$ for all $p=1,2,\cdots$ by Theorem 1. Since $E_{2l+1,2l} = E_{2l,2l}^{\perp}E_{2l+1,2l}\,E_{2l,2l}$ and $E_{2l+1,2l} = E_{2l+1,2l+1}E_{2l+1,2l}E_{2l+1,2l+1}^{\perp}$ ($l = 1,2,\cdots; l\neq i$ and $l\neq k$),

$$\left.\begin{cases} \varphi(E_{2l+1,2l}) & = \varphi(E_{2l,2l}^{\perp}E_{2l+1,2l}E_{2l,2l}) \\ & = \varphi(E_{2l,2l}^{\perp})\varphi(E_{2l+1,2l})\varphi(E_{2l,2l}) \\ & = E_{2l,2l}^{\perp}\varphi(E_{2l+1,2l})E_{2l,2l} \text{ and} \\ \varphi(E_{2l+1,2l}) & = \varphi(E_{2l+1,2l+1}E_{2l+1,2l}E_{2l+1,2l+1}^{\perp}) \\ & = \varphi(E_{2l+1,2l+1})\varphi(E_{2l+1,2l})\varphi(E_{2l+1,2l+1}^{\perp}) \\ & = E_{2l+1,2l+1}\varphi(E_{2l+1,2l})E_{2l+1,2l+1}^{\perp} \end{cases}\right\} \cdots (*)$$

Comparing components of the first equation of $(*)$ with those of second equation of $(*)$, we have $\varphi(E_{2l+1,2l}) = \alpha_{2l+1,2l}E_{2l+1,2l}$ for all $l=1,2,\cdots (l\neq i$ and $l\neq k)$. Since $E_{2l+1,2l}\neq 0$, $\alpha_{2l+1,2l}\neq 0$. Similarly we can prove that $\varphi(E_{2p+1,2p+2}) = \alpha_{2p+1,2p+2}E_{2p+1,2p+2}$ for all $p=1,2,\cdots$ and $\varphi(E_{12}) = \alpha_{l2}E_{12}$.

THEOREM 3. *Let $\varphi : \mathcal{A}^{(3)}_{\infty(i,k)} \longrightarrow \mathcal{A}^{(3)}_{\infty(i,k)}$ be an isomorphism such that $\varphi(E_{pp}) = E_{pp}$ for all $p=1,2,\cdots$ and let $\varphi(E_{2l+1,2l}) = \alpha_{2l+1,2l}$*

$E_{2l+1,2l}$, $\varphi(E_{2p+1,2p+2}) = \alpha_{2p+1,2p+2}E_{2p+1,2p+2}$ and $\varphi(E_{12}) = \alpha_{12}E_{12}$ $(l, p = 1, 2, \cdots; l \neq i$ and $l \neq k)$. Then there exists a diagonal linear transformation T such that $\varphi(A) = TAT^{-1}$ for all A in $\mathcal{A}^{(3)}_{\infty(i,k)}$.

Proof. Let $A = (a_{ij})$ be in $\mathcal{A}^{(3)}_{\infty(i,k)}$. Then $\varphi(A) = (\alpha_{ij}a_{ij})$ by Theorem 1. Let $T = (t_i)$ be a diagonal-matrix whose (i, i)-component is t_i and $t_i \neq 0$ for all $i = 1, 2, \cdots$. Then $TAT^{-1} = (t_i a_{ij} t_j^{-1})$. So if the linear system for unknown variables t_i $(i = 1, 2, \cdots)(*)$;

$$\begin{aligned}
\alpha_{12} &= t_1 t_2^{-1}, \\
\alpha_{32} &= t_3 t_2^{-1}, \\
\alpha_{34} &= t_3 t_4^{-1}, \\
&\vdots \\
\alpha_{2i-1,2i} &= t_{2i-1} t_{2i}^{-1}, \\
\alpha_{2i+1,2i+2} &= t_{2i+1} t_{2i+2}^{-1}, \\
\alpha_{2i+3,2i+2} &= t_{2i+3} t_{2i+2}^{-1}, \\
&\vdots \\
\alpha_{2k-1,2k-2} &= t_{2k-1} t_{2k-2}^{-1}, \\
\alpha_{2k-1,2k} &= t_{2k-1} t_{2k}^{-1}, \\
\alpha_{2k+1,2k+2} &= t_{2k+1} t_{2k+2}^{-1}, \\
\alpha_{2k+3,2k+2} &= t_{2k+3} t_{2k+2}^{-1}, \\
&\vdots
\end{aligned}$$

then $\varphi(A) = TAT^{-1}$ for all A in $\mathcal{A}^{(3)}_{\infty(i,k)}$. Put $t_1 = 1, t_{2i+1} = 1$ and $t_{2k+1} = 1$. Then

$$\begin{aligned}
t_2 &= \alpha_{12}^{-1} \\
t_3 &= \alpha_{32}\alpha_{12}^{-1} \\
t_4 &= \alpha_{34}^{-1}\alpha_{32}\alpha_{12}^{-1} \\
&\vdots
\end{aligned}$$

$$
\begin{aligned}
t_{2i} &= \alpha^{-1}_{2i-1,2i}\alpha_{2i-1,2i-2}\cdots\alpha_{32}\alpha^{-1}_{12}\\
t_{2i+2} &= \alpha^{-1}_{2i+1,2i+2}\\
t_{2i+3} &= \alpha_{2i+3,2i+2}\alpha^{-1}_{2i+1,2i+2}\\
t_{2i+4} &= \alpha^{-1}_{2i+3,2i+4}\alpha_{2i+3,2i+2}\alpha^{-1}_{2i+1,2i+2}\\
&\vdots
\end{aligned}
$$

$$
\begin{aligned}
t_{2k-1} &= \alpha_{2k-1,2k-2}\alpha^{-1}_{2k-3,2k-2}\alpha_{2k-3,2k-4}\cdots\alpha_{2i+3,2i+2}\alpha^{-1}_{2i+1,2i}\\
t_{2k} &= \alpha^{-1}_{2k-1,2k}\alpha_{2k-1,2k-2}\alpha^{-1}_{2k-3,2k-2}\alpha_{2k-3,2k-4}\\
&\qquad\cdots\alpha_{2i+3,2i+2}\alpha^{-1}_{2i+1,2i+2}\\
t_{2k+2} &= \alpha^{-1}_{2k+1,2k+2}\\
t_{2k+3} &= \alpha_{2k+3,2k+2}\alpha^{-1}_{2k+1,2k+2}\\
&\vdots
\end{aligned}
$$

Thus the linear system $(*)$ has solutions and hence $\varphi(A) = TAT^{-1}$ for all A in $\mathcal{A}^{(3)}_{\infty(i,k)}$ and T constructed the above.

THEOREM 4. (Gilfeather and Moore [9]) *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be commutative subspace lattices on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and suppose that $\varphi : Alg\mathcal{L}_1 \longrightarrow Alg\mathcal{L}_2$ is an algebraic isomorphism. Let $\mathcal{M}$ be a maximal abelian self-adjoint subalgebra(masa) contained in $Alg\mathcal{L}_1$. Then there exist a bounded invertible operator $Y : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ and an isomorphism $\rho : Alg\mathcal{L}_1 \longrightarrow Alg\mathcal{L}_2$ such that*
i) $\rho(M) = M$ *for all M in $\mathcal{M}$ and*
ii) $\varphi(A) = Y\rho(A)Y^{-1}$ *for all A in $Alg\mathcal{L}_1$.*

THEOREM 5. *Let $\varphi : \mathcal{A}^{(3)}_{\infty(i,k)} \longrightarrow \mathcal{A}^{(3)}_{\infty(i,k)}$ be an isomorphism. Then there exists an invertible linear transformation T from $\mathcal{H}$ onto $\mathcal{H}$ such that $\varphi(A) = TAT^{-1}$ for all A in $\mathcal{A}^{(3)}_{\infty(i,k)}$.*

Proof. Since $(\mathcal{A}^{(3)}_{\infty(i,k)}) \cap (\mathcal{A}^{(3)}_{\infty(i,k)})^*$ is a masa of $\mathcal{A}^{(3)}_{\infty(i,k)}$ and E_{ii} is in $(\mathcal{A}^{(3)}_{\infty(i,k)}) \cap (\mathcal{A}^{(3)}_{\infty(i,k)})^*$ for all $i = 1, 2, \cdots$, by Theorem 4 there exist a bounded invertible operator Y in $\mathcal{B}(H)$ and an isomorphism $\rho : \mathcal{A}^{(3)}_{\infty(i,k)} \longrightarrow \mathcal{A}^{(3)}_{\infty(i,k)}$ such that $\rho(E_{ii}) = E_{ii}$ for all $i = 1, 2, \cdots$ and $\varphi(A) = Y\rho(A)Y^{-1}$ for all A in $\mathcal{A}^{(3)}_{\infty(i,k)}$. By Theorem 3 $\rho(A) = SAS^{-1}$ for some diagonal linear transformation S and all A in $\mathcal{A}^{(3)}_{\infty(i,k)}$. Hence $\varphi(A) = Y\rho(A)Y^{-1} = (YS)A(S^{-1}Y^{-1})$ for all A in $\mathcal{A}^{(3)}_{\infty(i,k)}$. Let $T = YS$. Then $\varphi(A) = TAT^{-1}$ for all A in $\mathcal{A}^{(3)}_{\infty(i,k)}$.

THEOREM 6. *Let $\varphi : \mathcal{A}^{(3)}_{\infty(i,k)} \longrightarrow \mathcal{A}^{(3)}_{\infty(i,k)}$ be an isomorphism. Then there exists an invertible linear transformation T all of whose entries are zero except for the (p,p)-component, the $(2q\text{-}1, 2q)$-component and the $(2h+1, 2h)$-component ($h, p, q = 1, 2, \cdots, h \neq i$ and $h \neq k$) such that $\varphi(A) = TAT^{-1}$ for all A in $\mathcal{A}^{(3)}_{\infty(i,k)}$.*

Proof. Let $\varphi : \mathcal{A}^{(3)}_{\infty(i,k)} \longrightarrow \mathcal{A}^{(3)}_{\infty(i,k)}$ be an isomorphism. Then by Theorem 5 there exists an invertible linear transformation T such that $\varphi(A) = TAT^{-1}$ for all A in $\mathcal{A}^{(3)}_{\infty(i,k)}$. Let $A = (a_{ij})$ and $\varphi(A) = (b_{ij})$ be in $\mathcal{A}^{(3)}_{\infty(i,k)}$, and let $T = (t_{ij})$. Then $\varphi(A)T = TA \cdots (*)$.

(1) $t_{2n,2m-1} = 0$ for all n, m. First, we will show that $t_{2n,1} = 0$ for all n. Suppose that $t_{2n,1} \neq 0$ for some n. Comparing the $(2n,1)$-component of $\varphi(A)T$ with that of TA, $a_{11} = b_{2n,2n} \cdots (*_1)$. Comparing the $(2n,3)$-component of $\varphi(A)T$ with that of TA, $t_{2n,3}a_{33} = t_{2n,3}b_{2n,2n}$. So $t_{2n,3}(b_{2n,2n} - a_{33}) = t_{2n,3}(a_{11} - a_{33}) = 0$ by $(*_1)$. Since the equation $(*)$ holds for all A in $\mathcal{A}^{(3)}_{\infty(i,k)}$, $t_{2n,3} = 0$. Comparing the $(2n,2)$-component of $\varphi(A)T$ with that of TA, $b_{2n,2n}t_{2n,2} = t_{2n,1}a_{12} + t_{2n,2}a_{22}$ if $n = i$ and $b_{2n,2n}t_{2n,2} = t_{2n,1}a_{12} + t_{2n,2}a_{22} + t_{2n,3}a_{32}$ if $n \neq i$. Since $t_{2n,3} = 0$ and $a_{11} = b_{2n,2n}$, we have that $t_{2n,2}(a_{11} - a_{22}) = t_{2n,1}a_{12}$. Since the equation $(*)$ holds for all A in $\mathcal{A}^{(3)}_{\infty(i,k)}$, we have a contradiction if $a_{11} = a_{22}$ and $a_{12} \neq 0$. Thus $t_{2n,1} = 0$. Show that if $t_{2n,2l-1} = 0$, then $t_{2n,2l+1} = 0$ for all $l = 1, 2, \cdots$. Suppose that $t_{2n,2l+1} \neq 0$ for some l. Comparing the $(2n, 2l+1)$-component of $\varphi(A)T$ with that of TA, $a_{2l+1,2l+1} = b_{2n,2n}$. Comparing the $(2n, 2l)$-component of $\varphi(A)T$ with that of TA, $b_{2n,2n}t_{2n,2l} = t_{2n,2l-1}a_{2l-1,2l} + t_{2n,2l}a_{2l,2l} + t_{2n,2l+1}a_{2l+1,2l}$ if $l \neq i$ and $l \neq k$. Since $t_{2n,2l-1} = 0$ and $b_{2n,2n} = a_{2l+1,2l+1}$,

$t_{2n,2l}(a_{2l+1,2l+1} - a_{2l,2l}) = t_{2n,2l+1}a_{2l+1,2l}$. Comparing the $(2n, 2l + 2)$-component of $\varphi(A)T$ with that of TA, $t_{2n,2l+1}a_{2l+1,2l+2} + t_{2n,2l+2} a_{2l+2,2l+2} + t_{2n,2l+3} a_{2l+3,2l+2} = b_{2n,2n}t_{2n,2l+2}(l+1 \neq k)$ or $t_{2n,2l+1} a_{2l+1,2l+2} + t_{2n,2l+2}a_{2l+2,2l+2} = b_{2n,2n}t_{2n,2l+2}(l+1 \neq k)$ if $l = i$. Since $b_{2n,2n} = a_{2l+1,2l+1}$, $t_{2n,2l+1}a_{2l+1,2l+2}+t_{2n,2l+2}$ $(a_{2l+2,2l+2}- a_{2l+1,2l+1})$ $+t_{2n,2l+3}a_{2l+3,2l+2} = 0$ or $t_{2n,2l+1}a_{2l+1,2l+2} + t_{2n,2l+2}(a_{2l+2,2l+2}\text{-}a_{2l+1,2l+1}) = 0$. Since the equation $(*)$ holds for all A in $\mathcal{A}^{(3)}_{\infty(i,k)}$, we have a contradiction. Thus if $t_{2n,2l-1} = 0$, then $t_{2n,2l+1} = 0$ for all $l = 1, 2, \cdots$. Therefore $t_{2n,2m-1} = 0$ for all n, m.

(2) If $t_{2n,2m} \neq 0$, then $a_{2m,2m} = b_{2n,2n}$ for all n, m. For, comparing the $(2n, 2m)$-component of $\varphi(A)T$ with that of TA, $b_{2n,2n}t_{2n,2m} = t_{2n,2m-1}a_{2m-1,2m} + t_{2n,2m}a_{2m,2m} + t_{2n,2m+1}a_{2m+1,2m}$ if $m \neq i$ and $b_{2n,2n}t_{2n,2m} = t_{2n,2m-1}a_{2m-1,2m} + t_{2n,2m}a_{2m,2m}$ if $m = i$. Since $t_{2n,2m} \neq 0$, $t_{2n,2m-1} = 0$ and $t_{2n,2m+1} = 0$, $b_{2n,2n} = a_{2m,2m}$. Similarly, we can get the following. If $t_{2n-1,2m-1} \neq 0$, then $b_{2n-1,2n-1} = a_{2m-1,2m-1}$ for all n, m. If $t_{2n,2m} \neq 0$, then $t_{2n,l} = 0$ for all $l (l \neq 2m)$. If $t_{2n,2m} \neq 0$, then $t_{2l,2m} = 0$ for all $l (l \neq i)$. If $t_{2n-1,2m-1} \neq 0$, then $t_{l,2m-1} = 0$ for all $l(l \neq 2n-1)$. If $t_{2n-1,2m-1} \neq 0$, then $t_{2n-1,2l-1} = 0$ for all $l(l \neq m)$. If $t_{11} \neq 0$, $t_{2i+1,2i+1} \neq 0$ and $t_{2k+1,2k+1} \neq 0$, then T has the form of elements of $\mathcal{A}^{(3)}_{\infty(i,k)}$. For, let $t_{11} \neq 0$. Comparing the $(1,2)$-component of $\varphi(A)T$ with that of TA, $t_{22} \neq 0$. Suppose that $t_{11} \neq 0, t_{22} \neq 0, \cdots, t_{2l,2l} \neq 0$. Comparing the $(2l+1, 2l)$-component of $\varphi(A)T$ with that of TA ($l \neq i$ and $l \neq k$), $t_{2l+1,2l+1} \neq 0$. Suppose that $t_{11} \neq 0, t_{22} \neq 0, \cdots, t_{2l-1,2l-1} \neq 0$. Comparing the $(2l-1, 2l)$-component of $\varphi(A)T$ with that of TA ($l \neq i$ and $l \neq k$), $t_{2l,2l} \neq 0$. So by induction, $t_{ll} \neq 0$ for all $l = 1, 2, \cdots$. Similarly, we can get the following. If $t_{2n-1,2n-1} \neq 0$, $t_{2n,2n} \neq 0$ and $t_{2n+1,2n+1} \neq 0$, then $t_{2l-1,2n} = 0$ for $l \neq n$ and $l \neq n+1$. Finally, suppose that $t_{2i+1,2i} \neq 0$. Comparing the $(2i+1, 2i)$-component of $\varphi(A)T$ with that of TA, $b_{2i+1,2i}t_{2i,2i} + b_{2i+1,2i+1}t_{2i+1,2i} + b_{2i+1,2i+2}t_{2i+2,2i} = t_{2i+1,2i-1}a_{2i-1,2i} + t_{2i+1,2i}a_{2i,2i} + t_{2i+1,2i+1}a_{2i+1,2i}$. Since $a_{2i+1,2i} = 0, b_{2i+1,2i} = 0, t_{2i+2,2i} = 0, t_{2i+1,2i-1} = 0$ and $b_{2i+1,2i+1} = a_{2i+1,2i+1}$ by $(*_1)$, $t_{2i+1,2i}(a_{2i+1,2i+1} - a_{2i,2i}) = 0$. Since the equation $(*)$ holds for all A in $\mathcal{A}^{(3)}_{\infty(i,k)}$, we have a contradiction if $a_{2i+1,2i+1} \neq a_{2i,2i}$. Thus $t_{2i+1,2i} = 0$. Similarly, we can prove that $t_{2k+1,2k} = 0$. It is easily

verified that $t_{2l-1,1}$ and $t_{2l-2,2}$ cannot both be nonzero, and $t_{2l-1,1}$ and $t_{2l,2}$ cannot both be nonzero ($l \geq 2$). If $t_{11} = 0$, then $t_{2j-1,1} \neq 0$ for some $j (j = 1, 2, \cdots)$. Suppose that $t_{2j-2,2} = 0$ and $t_{2j,2} = 0$. Comparing the $(2j-1, 2)$-component of $\varphi(A)T$ with that of TA, we have $t_{2j-1,2}(a_{11} - a_{22}) = t_{2j-1,1}a_{12}$ which is a contradiction. Thus $t_{2j-2,2} \neq 0$ or $t_{2j,2} \neq 0$. But this contradicts the just above fact $\cdots$ $(*_2)$. By a simple but tedious calculation, it is verified that $t_{2l-1,2i+1}$ and $t_{2l-2,2i+2}$ cannot both be nonzero, and $t_{2l-1,2i+1}$ and $t_{2l,2i+2}$ cannot both be nonzero ($l \neq i+1$). Suppose that $t_{2i+1,2i+1} = 0$. Then $t_{1,2i+1} \neq 0$ or $t_{2l-1,2i+1} \neq 0$ for some l. Suppose that $t_{1,2i+1} \neq 0$ and $t_{2,2i+2} = 0$. Comparing the $(1, 2i+2)$-component of $\varphi(A)T$ with that of TA, $b_{11}t_{1,2i+2} + b_{12}t_{2,2i+2} = t_{1,2i+1}a_{2i+1,2i+2} + t_{1,2i+2}a_{2i+2,2i+2} + t_{1,2i+3}a_{2i+3,2i+2}$. Since $b_{11} = a_{11}$, $t_{1,2i+2}(a_{11} - a_{2i+2,2i+2}) = t_{1,2i+1} a_{2i+1,2i+2} + t_{1,2i+3}a_{2i+3,2i+2}$. Since the equation $(*)$ holds for all A in $\mathcal{A}^{(3)}_{\infty(i,k)}$, we have a contradiction. So $t_{2,2i+2} \neq 0$. It contradicts the just above fact. If $t_{2l-1,2i+1} \neq 0$ for some l, then with argument similar to $(*_2)$ we have a contradiction. Hence $t_{2i+1,2i+1} \neq 0$.

Similarly we can prove that $t_{2k+1,2k+1} \neq 0$. Hence $t_{ll} \neq 0$ for all l and T has the form of elements of $\mathcal{A}^{(3)}_{\infty(i,k)}$.

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