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THE IMAGE OF A CONTINUOUS STRONG HIGHER DERIVATION IS CONTAINED IN THE RADICAL

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Let A be a Banach algebra over the complex field. A linear map $D : A \to A$ is a derivation if $D(xy) = xD(y) + D(x)y$ for all $x, y \in A$. A sequence $\{H_0, H_1, \cdots, H_m\}$ (resp. $\{H_0, H_1, \cdots\}$) of linear operators on A is a *higher derivation of rank* m (resp. infinitely rank) if for each $n = 0, 1, 2, \cdots, m$ (resp. $n = 0, 1, 2, \cdots$) and any $x, y \in A$,

$$H_n(xy) = \sum_{i=0}^{n} H_i(x)H_{n-i}(y).$$

These equations are called the Leibnitz identities. A higher derivation $\{H_n\}$ of rank m is *strong* if H_0 is an identity operator, and *continuous* if H_n is continuous for each $n = 1, 2, \cdots, m$. Note that a strong higher derivation of rank 1 is a derivation. For definitions and elementary properties of Banach algebras we refer to [1]. We denote by $rad(A)$ the radical of a Banach algebra A. A strong higher derivation $\{H_n\}$ of rank m on a Banach algebra A *maps into its radical* if $H_i(A) \subseteq rad(A)$ for $1 \le i \le m$.

I. M. Singer and J. Wermer [5] proved that every continuous derivation on a commutative Banach algebra A maps into its radical. They conjectured that the assumption of continuity is unnecessary. This became known as the Singer-Wermer Conjecture and was proved in 1987 by M.P.Thomas [6]. Thus we see that for every commutative, semisimple Banach algebra A, there are no nonzero derivations $D : A \to A$. This fact sometimes be used to show that a certain algebra cannot be given a norm which makes it a Banach algebra : See [1,18.22]. F.

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Gulick [2], N. P. Jewell [3], and R. J. Roy [4] have shown that the automatic continuity of derivations on semi-simple Banach algebras can be extended to higher derivations.

In this paper, we show that every continuous strong higher derivation, as well as every derivation, on a commutative Banach algebra maps into its radical.

THEOREM 1. *Every continuous strong higher derivation of any rank on a commutative Banach algebra maps into its radical*

In order to prove Theorem 1 we need the following lemma.

LEMMA 1. *If a sequence $\{H_n\}$ is a strong higher derivation of rank m on a Banach algebra A, then for each $1 < k \leq m, 1 \leq n$ and $x_1, x_2, \cdots, x_n, x \in A$*

$$(1) \qquad H_k(x_1x_2\cdots x_n) = \sum_{\substack{a_1+a_2+\cdots+a_n=k \\ 0\leq a_i}} H_{a_1}(x_1)H_{a_2}(x_2)\cdots H_{a_n}(x_n)$$

(2)

$$H_k^n(x^n) = \sum_{\substack{a_{1,1}+a_{1,2}+\cdots+a_{1,n}=k \\ a_{2,1}+a_{2,2}+\cdots+a_{2,n}=k \\ \cdots \\ a_{n,1}+a_{n,2}+\cdots+a_{n,n}=k \\ 0\leq a_{i,j}}} H_{a_{n,1}}(H_{a_{n-1,1}}(\cdots(H_{a_{2,1}}(H_{a_{1,1}}(x)))\cdots))$$
$$\cdot H_{a_{n,2}}(H_{a_{n-1,2}}(\cdots(H_{a_{2,2}}(H_{a_{1,2}}(x)))\cdots))$$
$$\cdots H_{a_{n,n}}(H_{a_{n-1,n}}(\cdots(H_{a_{2,n}}(H_{a_{1,n}}(x)))\cdots)).$$

Proof. By definition, for each $n > 0$

$$H_k(x_1x_2\cdots x_n) = \sum_{i_1=0}^{k} H_{i_1}(x_1)H_{k-i_1}(x_2x_3\cdots x_n).$$

If $n > 2$, we substitute $H_{k-i_1}(x_2x_3\cdots x_n)$ in the recursion formula. This procedure give us the sum

$$H_k(x_1x_2\cdots x_n) = \sum_{i_1=0}^{k}\sum_{i_2=0}^{k-i_1}\sum_{i_3=0}^{k-i_1-i_2}\cdots$$
$$\sum_{i_{n-1}=0}^{k-i_1-i_2-\cdots-i_{n-2}} H_{i_1}(x_1)H_{i_2}(x_2)\cdots H_{k-i_1-i_2-\cdots-i_{n-1}}(x_n).$$

Thus for each $1 < k \leq m$ and $x_1, x_2, \cdots, x_n \in A$

$$H_k(x_1 x_2 \cdots x_n) = \sum_{\substack{a_1+a_2+\cdots+a_n=k \\ 0 \leq a_i}} H_{a_1}(x_1) H_{a_2}(x_2) \cdots H_{a_n}(x_n).$$

Also we have

$$\begin{aligned} H_k^2(x^n) &= \sum_{\substack{a_1+a_2+\cdots+a_n=k \\ 0 \leq a_i}} H_k(H_{a_1}(x) H_{a_2}(x) \cdots H_{a_n}(x)) \\ &= \sum_{\substack{a_1+a_2+\cdots+a_n=k \\ b_1+b_2+\cdots+b_n=k \\ 0 \leq a_i}} H_{b_1}(H_{a_1}(x)) H_{b_2}(H_{a_2}(x)) \cdots H_{b_n}(H_{a_n}(x)). \end{aligned}$$

By the recursion formula, we obtain the equation (2).

Proof of Theorem 1. Let $\{H_n\}$ be a continuous strong higher derivation of rank m on a commutative Banach algebra A. By Singer-Wermer's theorem, $H_1(A) \subseteq rad(A)$.

It suffices to prove the case $k = 2$. Let P be a primitive ideal of $A, y \in A$ and $x \in P$. Then

$$yH_2(x) = H_2(yx) - H_2(y)x - H_1(y)H_1(x) \in H_2(P) + P.$$

This shows that $(H_2(P) + P)/P$ is a left ideal of A/P. A similar argument shows that it is a right ideal and we conclude that $(H_2(P) + P)/P$ is an ideal of A/P. For $n > 2$ and $0 \leq a_{i,j}$ $(i, j = 1, 2, \cdots, n)$, let

$$\begin{aligned} a_{1,1} + a_{1,2} + \cdots + a_{1,n} &= 2, \\ a_{2,1} + a_{2,2} + \cdots + a_{2,n} &= 2, \\ &\cdots \\ a_{n,1} + a_{n,2} + \cdots + a_{n,n} &= 2. \end{aligned}$$

If $a_{i,j} = 1$ for some $i, j = 1, 2, \cdots, n$, then for all $n > 2$

$$(*) \qquad \begin{aligned} & H_{a_{n,1}}(H_{a_{n-1,1}}(\cdots(H_{a_{2,1}}(H_{a_{1,1}}(x)))\cdots)) \\ & \cdot H_{a_{n,2}}(H_{a_{n-1,2}}(\cdots(H_{a_{2,2}}(H_{a_{1,2}}(x)))\cdots)) \\ & \cdots\cdots H_{a_{n,n}}(H_{a_{n-1,n}}(\cdots(H_{a_{2,n}}(H_{a_{1,n}}(x)))\cdots)) \end{aligned}$$

is an element of P. If $a_{i,j} \neq 1$ for all $i, j = 1, 2, \cdots, n$, then for all $n > 2$

$$(*) = (H_2(x))^n.$$

Thus we have

$$H_2^n(x^n) \in n!(H_2(x))^n + P$$

for all $n > 2$. Then

$$(n!)^{\frac{1}{n}} ||(Q_P H_2(x))^n||^{\frac{1}{n}} = ||Q_P H_2^n(x^n)||^{\frac{1}{n}} \leq ||H_2|| ||x^n||^{\frac{1}{n}},$$

where $Q_P : A \to A/P$ is a natural quotient map. Since $\|H_2\| \|x^n\|^{\frac{1}{n}}$ is bounded and $(n!)^{\frac{1}{n}} \to \infty$, this shows that $Q_P H_2(x)$ is quasi-nilpotent. Since x was an arbitrary element of P and A/P is semisimple,

$$(H_2(P) + P)/P \subseteq rad(A/P) = 0.$$

Thus $H_2(P) \subseteq P$, and so $H_2(A) \subseteq rad(A)$. The proof of the theorem is complete.

References

1. F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, (Springer-Verag, Berlin, 1973).
2. F. Gulick, *Systems of derivations*, Trans. Amer. Math. Soc. **149** (1970), 465-488.
3. N. P. Jewell, *Continuity of Module and Higher Derivations*, 68 (1977), 91-98.
4. R. J. Loy, Continuity of Higher Derivations, Proc. Amer. Math. Soc. **37** (1973), 505-510.
5. I. M. Singer and J. Wermer, *Derivations on Commutative Normed Algebras*, Math. Ann. **129** (1955), 260-264.
6. M. P. Thomas, *The Image of a Derivation is contained in the Radical*, Ann. of Math. **128** (1988), 435-460.

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