Bull. Korean Math. Soc. **33** (1996), No. 2, pp. 221–228

A NOTE ON JORDAN LEFT DERIVATIONS

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1. Introduction

Throughout, R will represent an associative ring with center $Z(R)$. A module X is said to be n-torsionfree, where n is an integer, if $nx = 0$, $x \in X$ implies $x = 0$. An additive mapping $D : R \longrightarrow X$, where X is a left R-module, will be called a *Jordan left derivation* if $D(a^2) = 2aD(a)$, $a \in R$. M. Brešar and J. Vukman [1] showed that the existence of a nonzero Jordan left derivation of R into X implies R is commutative if X is a 2-torsionfree and 3-torsionfree left R-module. They conjectured that in their results the assumption that X is 3-torsionfree can be avoided. We prove that the result holds without this requirement.

2. Left Jordan Derivations

The following proposition is due to M. Brešar and J. Vukman [1].

PROPOSITION 2.1. *Let R be a ring and X be a 2-torsionfree left R-module. If $D : R \longrightarrow X$ is a Jordan left derivation then for all $a, b, c \in R$:*

(i) $D(ab + ba) = 2aD(b) + 2bD(a)$,
(ii) $D(aba) = a^2D(b) + 3abD(a) - baD(a)$,
(ii) $D(abc + cba) = (ac + ca)D(b) + 3abD(c) + 3cbD(a) - baD(c) - bcD(a)$,
(iv) $(ab - ba)aD(a) = a(ab - ba)D(a)$,
(v) $(ab - ba)\big(D(ab) - aD(b) - bD(a)\big) = 0$.

Received November 5, 1994. Revised September 4, 1995.
1991 AMS Subject Classification: 16W25.
Key words and phrases: n-torsionfree, left R-module, Jordan left derivation.
*This paper was supported in part by NON DIRECTED RESEARCH FUND, Korea Research Foundation, 1994.

We need the other fundamental results to prove the main theorem.

PROPOSITION 2.2. *Let R be a ring and X be a 2-torsionfree left R-module. If $D : R \longrightarrow X$ is a Jordan left derivation then for all $a, b \in R$:*

(i) $D(a^2b) = a^2D(b) + (ab + ba)D(a) + aD(ab - ba)$,
(ii) $D(ba^2) = a^2D(b) + (3ba - ab)D(a) - aD(ab - ba)$,
(iii) $(ab - ba)D(ab - ba) = 0$,
(iv) $(a^2b - 2aba + ba^2)D(b) = 0$.

Proof. Throughout the proof, let a, b be arbitrary elements in R.
(i) It follows from Proposition 2.1 (i) that

$$D(aba + ba^2) = 2(aD(ba) + baD(a)), \tag{1}$$

$$D(a^2b + aba) = 2(aD(ab) + abD(a)). \tag{2}$$

Taking (2) minus (1), we see that

$$D(a^2b + ba^2) = 2(aD(ab - ba) + (ab - ba)D(a)). \tag{3}$$

Replacing a^2 for a in Proposition 2.1 (i), we have

$$D(a^2b + ba^2) = 2(a^2D(b) + 2baD(a)). \tag{4}$$

Hence, taking (3) plus (4), and then using the assumption that X is 2-torsionfree, we obtain

$$D(a^2b) = a^2D(b) + (ab + ba)D(a) + aD(ab - ba).$$

(ii) As in the proof of the Case (i) taking (4) minus (3), we have

$$D(ba^2) = a^2D(b) + (3ba - ab)D(a) - aD(ab - ba).$$

(iii) From Proposition 2.1 (v),

$$(ab - ba)(D(ab) - aD(b) - bD(a)) = 0. \tag{5}$$

Combining Proposition 2.1 (i) and (v),

$$(ab - ba)(D(ba) - aD(b) - bD(a)) = 0. \tag{6}$$

Taking (5) minus (6),

$$(ab - ba)D(ab - ba) = 0.$$

(iv) Applying Proposition 2.2 (i) and (ii), we have $(ab - ba)D(ab - ba) = 0$ $a, b \in R$. And so,

$$\begin{aligned} D((ab - ba)^2) &= D(a(bab) + (bab)a) - D(ab^2a) - D(ba^2b) \\ &= 2aD(bab) + 2babD(a) - D(ab^2a) - D(ba^2b) \\ &= -3(a^2b - 2aba + ba^2)D(b) - (b^2a - 2bab + ab^2)D(a). \end{aligned}$$

On the other hand, we have $D((ab - ba)^2) = 2(ab - ba)D(ab - ba) = 0$. Consequently, we get

$$3(a^2b - 2aba + ba^2)D(b) + (b^2a - 2bab + ab^2)D(a) = 0. \tag{7}$$

From Proposition 2.1 (iv)

$$(a^2b - 2aba + ba^2)D(a) = 0. \tag{8}$$

Replacing $a + b$ for a in (8) ,

$$(a + b)(ab - ba)(D(a) + D(b)) - (ab - ba)(a + b)(D(a) + D(b)) = 0. \tag{9}$$

Hence it follows from (8) and (9) that

$$(a^2b - 2aba + ba^2)D(b) - (b^2a - 2bab + ab^2)D(a) = 0. \tag{10}$$

Taking (7) plus (10), and then using the assumption that X is 2-torsionfree we obtain

$$(a^2b - 2aba + ba^2)D(b) = 0. \tag{11}$$

Hence from (10) we get

$$(b^2a - 2bab + ab^2)D(a) = 0.$$

3. Main Theorem

THEOREM3.1. *Let R be a ring and X be a 2-torsionfree left R-module. Suppose that $aRx = 0$ with $a \in R, x \in X$ implies that either $a = 0$ or $x = 0$. If there exists a nonzero Jordan let derivation $DR \longrightarrow X$ then R is commutative.*

Prof. Form Proposition 2.1 (iv), we have

$$(x^2y - 2xyx + yx^2)D(x) = 0 \text{ for all } x, y \in R.$$

Substituting $ab - ba$ for x, we have

(12)
$$\begin{aligned}(ab-ba)^2yD(ab-ba) - 2(ab-ba)y(ab-ba)D(ab-ba)\\ + y(ab-ba)^2D(ab-ba) = 0 \text{ for all } a, b, y \in R.\end{aligned}$$

Using Proposition 2.2 (iii) we get

$$(ab-ba)^2yD(ab-ba) = 0 \text{ for all } a, b, y \in R. \tag{13}$$

From the assumption, either $(ab-ba)^2 = 0$ or $D(ab-ba) = 0$ for all $a, b \in R$.

Suppose that $(ab-ba)^2 = 0$ for all $a, b \in R$. Applying Proposition 2.1 (i), (ii) and Proposition 2.2 (iii) we obtain

$$\begin{aligned}E &= D\big(((ab-ba)x)(ab-ba)y(ab-ba)\\ &\quad + (ab-ba)y(ab-ba)((ab-ba)x)\big)\\ &= 2\{(ab-ba)xD\big((ab-ba)y(ab-ba)\big)\\ &\quad + (ab-ba)y(ab-ba)D\big((ab-ba)x\big)\}\\ &= 6(ab-ba)x(ab-ba)yD(ab-ba)\\ &\quad + (ab-ba)y\{2(ab-ba)D\big((ab-ba)x\big)\}.\end{aligned} \tag{14}$$

On the other hand,

$$\begin{aligned}E &= D\big((ab-ba)(x(ab-ba)y)(ab-ba)\big)\\ &= 3(ab-ba)x(ab-ba)yD(ab-ba).\end{aligned} \tag{15}$$

Comparing (14) and (15), we arrive at

$$(16)\quad 3(ab-ba)x(ab-ba)yD(ab-ba) + (ab-ba)y\{2(ab-ba)D(ab-ba)x)\} = 0 \text{ for all } a,b,x,y\in R.$$

And,

$$(17)\quad \begin{aligned} F &= D\big((ab-ba)x(ab-ba) + x(ab-ba)(ab-ba)\big) \\ &= D\big((ab-ba)x(ab-ba)\big) \\ &= 3(ab-ba)xD(ab-ba). \end{aligned}$$

On the other hand, we also have

$$(18)\quad \begin{aligned} F &= 2\{(ab-ba)D\big(x(ab-ba)\big) + x(ab-ba)D(ab-ba)\} \\ &= 2(ab-ba)D\big(x(ab-ba)\big). \end{aligned}$$

Comparing (7) and (18) we get

$$(19)\quad 2(ab-ba)D\big(x(ab-ba)\big) = 3(ab-ba)xD(ab-ba) \quad \text{for all } a,b,x\in R.$$

Using Proposition 2.1 (i) and the assumption that $(ab-ba)^2=0$ for all $a,b\in R$ we have

$$(20)\quad \begin{aligned} (ab-ba)D(x(ab-ba)+(ab-ba)x) &= 2(ab-ba)^2Dx + 2(ab-ba)xD(ab-ba) \\ &= 2(ab-ba)xD(ab-ba) \text{ for all } a,b,x\in R. \end{aligned}$$

From (19) and (20) we obtain

$$(21)\quad 3(ab-ba)\{D(x(ab-ba)) + D((ab-ba)x)\} = 4(ab-ba)D(x(ab-ba)) \text{ for all } a,b,x\in R.$$

Thus

(22)
$$(ab-ba)D(x(ab-ba)) = 3(ab-ba)D((ab-ba)x) \text{ for all } a,b,x \in R.$$

From (22), we get

(23)
$$\begin{aligned}(ab-ba)&D(x(ab-ba)) + (ab-ba)x) \\ &= 3(ab-ba)D((ab-ba)x) + (ab-ba)D((ab-ba)x) \\ &= 4(ab-ba)D((ab-ba)x) \text{ for all } a,b,x \in R.\end{aligned}$$

And so, one obtains

(24)
$$\begin{aligned}&(ab-ba)D(x(ab-ba) + (ab-ba)x) \\ &\qquad = 2(ab-ba)\{xD(ab-ba) + (ab-ba)Dx\} \text{ for all } a,b,x \in R.\end{aligned}$$

From (23) and (24), we have

(25)
$$\begin{aligned}&2\{2(ab-ba)D((ab-ba)x) - (ab-ba)xD(ab-ba) - (ab-ba)^2Dx\} \\ &\qquad = 0 \text{ for all } a,b,x \in R.\end{aligned}$$

We have assumed that X is 2-torsionfree, and so

(26)
$$2(ab-ba)D((ab-ba)x) = (ab-ba)xD(ab-ba) \text{ for all } a,b,x \in R.$$

Thus from (16) and (26) it follows that

(27)
$$\begin{aligned}&3(ab-ba)x(ab-ba)yD(ab-ba) + (ab-ba)y(ab-ba)xD(ab-ba) \\ &\qquad = 0 \text{ for all } a,b,x \in R.\end{aligned}$$

Replacing $y(ab-ba)y$ for x in (26), we have

$$\begin{aligned}&2(ab-ba)D((ab-ba)y(ab-ba)y) = (ab-ba)y(ab-ba)yD(ab-ba) \\ &\qquad \text{for all } a,b,x,y \in R.\end{aligned}$$

Thus,

(28)
$$4(ab-ba)^2yD((ab-ba)y) = (ab-ba)y(ab-ba)yD(ab-ba)$$
$$\text{for all } a,b,y \in R.$$

We have assumed $(ab-ba)^2 = 0$for all $a,b \in R$ and so, one obtains

(29) $$(ab-ba)y(ab-ba)yD(ab-ba) = 0 \quad \text{for all } y \in R.$$

Replacing $x+y$ for y in (29) and using (29) again it follows that

(30)
$$(ab-ba)x(ab-ba)yD(ab-ba) + (ab-ba)y(ab-ba)xD(ab-ba)$$
$$= 0. \quad \text{for all} \quad x,y \in R.$$

Thus from (27) and (30)

(31) $$(ab-ba)x(ab-ba)yD(ab-ba) = 0 \quad \text{for all} \quad x,y \in R.$$

From (31) it follows that for each $a \in R$ either $a \in Z(R)$ or $D(ab-ba) = 0$ for all $b \in R$. Hence we consider the case $D(ab-ba) = 0$ for all $b \in R$. So we have

$$\begin{aligned} 2D((ba)a) &= D((ba)a + a(ba)) \\ &= 2\{a^2D(b) + abD(a) + baD(a)\}. \end{aligned}$$

Since X is 2-torsionfree, we obtain

(32) $$D((ba)a) = a^2D(b) + abD(a) + baD(a).$$

Using Proposition 2.2 (ii) and (32), we get

(33) $$(ab-ba)D(a) = 0 \qquad \text{for all} \quad b \in R.$$

Replacing bx for b in (33), we have

(34)
$$\begin{aligned} 0 &= (abx - bxa)D(a) \\ &= (abx - bax + bax - bxa)D(a) \\ &= (ab-ba)xDa + b(ax - xa)D(a) \\ &= (ab-ba)xD(a) \end{aligned}$$

Thus

$$(ab-ba)xD(a) = 0 \quad \text{for all} \quad a,b,x \in R.$$

Therefore it follows that for each $a \in R$ either $a \in Z(R)$ or $D(a) = 0$. Since $D \neq 0$, R is commutative.

References

1. M. Brešar and J. Vukman, *On left derivations and related mappings*, Proc. Amer. Math. Soc. **10** (1990), 7-16.

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