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# REPRESENTATIONS OF INVOLUTIVE SEMIGROUPS

YOUNKI CHAE AND KEUNBAE CHOI

In this note we study the representations on Hilbert space of involutive semigroups, i.e., semigroups endowed with an involutive antiautomorphism. This subject is studied by K. H. Neeb and some interesting results are investigated ([3]).

An *involutive semigroup* is a semigroup $S$ together with an unary operation $* : S \to S$, $s \mapsto s^*$ satisfying $(s^*)^* = s$ and $(st)^* = t^*s^*$ for all $s, t \in S$. Throughout the remainder of this paper, we write $B(\mathcal{H})$ for the algebra of bounded operators on a Hilbert space $\mathcal{H}$ and we denote $T^*$ for the adjoint operator of $T \in B(\mathcal{H})$. Note that for $T \in B(\mathcal{H})$, $T^*$ is a bounded operator on $\mathcal{H}$ and is uniquely determined, and hence $B(\mathcal{H})$ is an involutive semigroup under $T \mapsto T^*$.

DEFINITION 1. Let $\mathcal{H}$ be a Hilbert space. A (*symmetric*) *representation* of an involutive semigroup $S$ in $\mathcal{H}$ is a homomorphism $\pi : S \to B(\mathcal{H})$ with the property that $\pi(s^*) = \pi(s)^*$. We say that $\pi$ is *non-degenerate* if $\pi(S)\mathcal{H}$ spans a dense subspace of $\mathcal{H}$ and that $\pi$ is *cyclic* if there exists an element $v \in \mathcal{H}$ such that $\pi(S)v$ spans a dense subspace of $\mathcal{H}$. In this case $v$ is called a *cyclic vector* for $\pi$.

DEFINITION 2. Let $S$ be an involutive semigroup, $\mathcal{H}$ be a Hilbert space, and let $\pi$ be a representation of $S$ in $\mathcal{H}$. Then $\pi$ is said to be *topologically irreducible* if $\mathcal{H} \neq \{0\}$ and if the only closed subspaces of $\mathcal{H}$ invariant under $\pi(S)$ are $\{0\}$ and $\mathcal{H}$.

LEMMA 3. *Let $S$ be an involutive semigroup, $\mathcal{H}$ be a Hilbert space, and let $\pi$ be a representation of $S$ in $\mathcal{H}$. Then*

(1) *A closed subspace $\mathcal{K} \subseteq \mathcal{H}$ is invariant under $\pi(S)$ if and only if its orthogonal complement $\mathcal{K}^\perp$ is invariant under $\pi(S)$.*

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(2) *If $J$ is an involutive ideal of $S$, then the closure of the subspace spanned by $\pi(J)\mathcal{H}$ is invariant under $\pi(S)$.*
(3) *If $I$ is an involutive ideal of $S$, then the Rees quotient semigroup $S/I$ is an involutive semigroup and $\pi$ naturally induces the representation $\widetilde{\pi}$ of $S/I$ in $\mathcal{H}$.*
(4) *The set $I = \pi^{-1}(0)$ is an involutive ideal of $S$, where $0$ is the zero operator on $\mathcal{H}$.*

*Proof.* (1) This is immediate from the fact that $\pi(S) = \pi(S)^*([3])$.

(2) Using the continuity and linearity of $\pi$, we can easily complete the proof.

(3) This is straightforward.

(4) Let $s \in S$ and let $a \in I$. Then $\pi(sa) = \pi(s) \circ \pi(a) = 0$ and $sa \in I$. □

For convenience, throughout this note the subspace spanned by a set $W$ of vectors will be denoted by span $W$.

PROPOSITION 4. *Let $S$ be an involutive semigroup with an involutive ideal, $\mathcal{H}$ be a Hilbert space, and let $\pi$ be a representation of $S$ in $\mathcal{H}$. Then $\pi$ is a direct sum of a non-degenerate representation and some representation of $S$.*

*Proof.* Let $J$ be an involutive ideal of $S$ and let $\mathcal{H}_J = \overline{\text{span } \pi(J)\mathcal{H}}$. We define two representations

$$\pi_J : S \to B(\mathcal{H}_J) \text{ by } \pi_J(x) = \pi(x)|_{\mathcal{H}_J}$$

and

$$\sigma_J : S \to B(\mathcal{H}_J^{\perp}) \text{ by } \sigma_J(x) = \pi(x)|_{\mathcal{H}_J^{\perp}}.$$

Then since $\mathcal{H} = \mathcal{H}_J \oplus \mathcal{H}_J^{\perp}$ and since $\mathcal{H}_J$ and its othogonal complement $\mathcal{H}_J^{\perp}$ are invariant subspaces of $\mathcal{H}$ under $\pi(S)$, $\pi = \pi_J \oplus \sigma_J$. In this case $\pi_J$ is non-degenerate representation of $S$. □

The following result is some characterization theorem of topologically irreducible representations of involutive semigroups and this result is very useful in our approaches. Let $\pi$ be a representation of an involutive semigroup $S$ in a Hilbert space $\mathcal{H}$. The dimension of $\mathcal{H}$ is called the dimension of $\pi$ and is denoted by dim $\pi$.

THEOREM 5. *Let $S$ be an involutive semigroup, $\mathcal{H}$ be a Hilbert space, and let $\pi$ be a representation of $S$ in $\mathcal{H}$. Then $\pi$ is topologically irreducible if and only if either every non-zero vector in $\mathcal{H}$ is cyclic for $\pi$ or dim $\pi = 1$.*

*Proof.* Suppose that $\pi$ is topologically irreducible. Let $v$ be a non-zero vector in $\mathcal{H}$. If $\overline{\text{span } \pi(S)v} \neq \mathcal{H}$, then it is a closed, proper, and invariant subspace of $\mathcal{H}$ under $\pi(S)$. Thus span $\pi(S)v = \{0\}$ and hence $\pi(S)v = \{0\}$. It follows that $\mathbb{C}v$ is non-zero closed invariant subspace of $\mathcal{H}$ under $\pi(S)$. Since $\pi$ is topologically irreducible, $\mathcal{H} = \mathbb{C}v$ and $\pi$ is 1-dimensional.

Conversely, let $\mathcal{K}$ be a non-zero closed invariant subspace of $\mathcal{H}$ under $\pi(S)$. If dim $\pi = 1$, then clearly $\mathcal{K} = \mathcal{H}$. Now suppose that every non-zero vector in $\mathcal{H}$ is cyclic for $\pi$ and let $v$ be a non-zero vector contained in $\mathcal{K}$. Then span $\pi(S)v \subseteq \mathcal{K}$ and its closure is $\mathcal{H}$. Thus $\mathcal{K} = \mathcal{H}$. □

THEOREM 6. *Let $S$ be an involutive semigroup, let $\mathcal{H}$ be a Hilbert space, let $\pi$ be a topologically irreducible representation of $S$ in $\mathcal{H}$, and let $J$ be an involutive ideal of $S$ such that $\pi(J) \neq \{0\}$. Then $\pi|_J$ is a topologically irreducible representation of $J$ in $\mathcal{H}$.*

*Proof.* Let $\pi : S \to B(\mathcal{H})$ be a topologically irreducible representation such that $\pi(J) \neq \{0\}$. We want to prove that $\overline{\text{span } \pi(J)\xi} = \mathcal{H}$ for every non-zero vector in $\mathcal{H}$. Since $\overline{\text{span } \pi(J)\xi}$ is invariant under $\pi(S)$, it must be $\mathcal{H}$ or $\{0\}$. If $\overline{\text{span } \pi(J)\xi} = \{0\}$, then for all $\eta \in$ span $\pi(J)\mathcal{H}$,

$$\begin{aligned} < \xi, \eta > &=< \xi, \sum_{k=1}^{n} \pi(j_k)h_k >= \sum_{k=1}^{n} < \xi, \pi(j_k)h_k > \\ &= \sum_{k=1}^{n} < \pi(j_k)^*\xi, h_k >= \sum_{k=1}^{n} < \pi(j_k^*)\xi, h_k > \\ &=< \sum_{k=1}^{n} \pi(j_k^*)\xi, h_k >= 0 \end{aligned}$$

Thus the vector $\xi$ is orthogonal to the subspace span $\pi(J)\mathcal{H}$ and so is orthogonal to $\overline{\text{span } \pi(J)\mathcal{H}}$ by the continuity of an inner product on $\mathcal{H}$. It follows that $\overline{\text{span } \pi(J)\mathcal{H}}$ is a proper subspace of $\mathcal{H}$. But since

$\overline{\text{span } \pi(J)\mathcal{H}}$ is invariant under $\pi(S)$, $\overline{\text{span } \pi(J)\mathcal{H}} = \{0\}$. Hence $\pi(J) = \{0\}$, which is a contradiction. This completes the proof. □

COROLLARY 7. *Let $\pi$ be a topologically irreducible representation of an involutive semigroup $S$ in $\mathcal{H}$ and let $I = \pi^{-1}(0)$. Then for each non-trivial involutive ideal $J$ of $S/I$, $\widetilde{\pi}|_J$ is a topologically irreducible representation of $J$ in $\mathcal{H}$.*

*Proof.* By the definition of Rees quotient semigroup, $S/I = \{\{a\} : a \in S \setminus I\} \cup \{I\}$. Thus in case that $I = \pi^{-1}(0)$, we can easily prove that if $\pi$ is a topologically irreducible representation of $S$ in $\mathcal{H}$ then $\widetilde{\pi}$ is a topologically irreducible representation of an involutive semigroup $S/I$ in $\mathcal{H}$. That completes the proof. □

Let $S_1$ and $S_2$ be two involutive semigroups, and let $S_1 \times S_2$ be their cartesian product. Then $S_1 \times S_2$ with coordinatewise multiplication and with coordinatewise involution is also an involutive semigroup. Let $\mathcal{H}_i$ be a Hilbert space with inner product $< , >_i$ for each $i = 1, 2$. Then there exists unique inner product $< , >$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ such that

$$< x_1 \otimes x_2,\ y_1 \otimes y_2 > = < x_1,\ y_1 >_1 < x_2,\ y_2 >_2$$

for all $x_1 \otimes x_2, y_1 \otimes y_2 \in \mathcal{H}_1 \otimes \mathcal{H}_2$. The metric completion of the inner product space $\mathcal{H}_1 \otimes \mathcal{H}_2$ is called the Hilbert space *tensor* product of $\mathcal{H}_1$ and $\mathcal{H}_2$ and is denoted by $\mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2$. We note that a bounded operator $T_1 \otimes T_2$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ induces naturally a bounded operator $\mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2$ and that

$$(T_1 \widehat{\otimes} T_2)^* = T_1^{*_1} \widehat{\otimes} T_2^{*_2} \text{ and } (S_1 \widehat{\otimes} S_2) \circ (T_1 \widehat{\otimes} T_2) = (S_1 \circ T_1) \widehat{\otimes} (S_2 \circ T_2)$$

where $*_i$ is an involution in $B(\mathcal{H}_i)$ for each $i = 1, 2$.

Now let $\pi_1 : S_1 \to B(\mathcal{H}_1)$ and $\pi_2 : S_2 \to B(\mathcal{H}_2)$ be representations of $S_1$ and $S_2$ respectively. We define a map $\pi_1 \otimes \pi_2 : S_1 \otimes S_2 \to B(\mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2)$ by

$$(\pi_1 \otimes \pi_2)(s_1,\ s_2) = \pi_1(s_1) \widehat{\otimes} \pi_2(s_2)$$

for all $(s_1,\ s_2) \in S_1 \times S_2$. Then we can easily prove that $\pi_1 \otimes \pi_2$ is a representation of $S_1 \times S_2$ in $\mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2$.

THEOREM 8. *Let $S_1$ and $S_2$ be involutive semigroups, and let $\pi_1$ and $\pi_2$ be cyclic representations of $S_1$ and $S_2$ in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Then $\pi_1 \otimes \pi_2$ is a cyclic representation of the product involutive semigroup $S_1 \times S_2$ in $\mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2$.*

*Proof.* Let $v_1$ and $v_2$ be vectors in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. We note that for $a \in \text{span } \pi_1(S_1)v_1$ and $b \in \text{span } \pi_2(S_2)v_2$,

$$\begin{aligned} a \otimes b &= \left(\sum_{k=1}^{n} \pi_1(s_{1_k})v_1\right) \otimes \left(\sum_{k=1}^{m} \pi_2(s_{2_k})v_2\right) \\ &= \sum_{j=1}^{m}\sum_{k=1}^{n} \left(\pi_1(s_{1_k})v_1 \otimes \pi_2(\pi_2(s_{2_j})v_2\right) \\ &= \sum_{(j,\ k)} \left(\pi_1(s_{1_k})v_1 \otimes \pi_2(\pi_2(s_{2_j})v_2\right) \end{aligned}$$

and that if $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are sequences in span $\pi_1(S_1)v_1$ and span $\pi_2(S_2)v_2$ respectively and if two sequences converge to $a$ and $b$ respectively, then we can easily show that the sequence $\{a_n \otimes b_n\}_{n=1}^{\infty}$ converges to $a \otimes b$ by using the linearity and continuity of an inner product and using the fact $< , >_{\mathcal{H}_1 \otimes \mathcal{H}_1} = < , >_{\mathcal{H}_1} < , >_{\mathcal{H}_2}$.

From the above note, we have that

$$\begin{aligned} \overline{\text{span } \pi_1(S_1)v_1} \otimes \overline{\text{span } \pi_2(S_2)v_2} &\subseteq \overline{\overline{\text{span } \pi_1(S_1)v_1} \otimes \overline{\text{span } \pi_2(S_2)v_2}} \\ &= \overline{\text{span}(\pi_1(S_1)v_1 \otimes \pi_2(S_2)v_2)} \end{aligned}$$

Thus if $v_1$ and $v_2$ are cyclic vectors for $\pi_1$ and $\pi_2$ respectively, then $v_1 \otimes v_2$ is a cyclic vector for $\pi_1 \otimes \pi_2$ because

$$\text{span}(\pi_1 \otimes \pi_2)(S_1 \times S_2)v_1 \otimes v_2 = \text{span}(\pi_1(S_1)v_1 \otimes \pi_2(S_2)v_2).$$

and $\mathcal{H}_1 \otimes \mathcal{H}_2$ is a dense subspace of $\mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2$. That completes the proof. □

We now study topologically irreducible representations of product involutive semigroups.

THEOREM 9. *Let $S_1$ and $S_2$ be involutive semigroups, and let $\pi_1$ and $\pi_2$ be topologically irreducible representations of $S_1$ and $S_2$ in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively such that dim $\pi_1 > 1$ and dim $\pi_2 > 1$ and that $\pi_1^{-1}(0) \neq \emptyset$ or $\pi_2^{-1}(0) \neq \emptyset$. Then every non-zero vector in $\mathcal{H}_1 \otimes \mathcal{H}_2$ is cyclic for $\pi_1 \otimes \pi_2$.*

*Proof.* Let $v$ is a non-zero vector in $\mathcal{H}_1 \otimes \mathcal{H}_2$. Then $v = \sum_{i=1}^{n} h_{1_i} \otimes h_{2_i}$. Since $\pi_1^{-1}(0) \neq \emptyset$ or $\pi_2^{-1}(0) \neq \emptyset$,

$$\begin{aligned}\pi_1 \otimes \pi_2(S_1 \times S_2)(v) &= \pi_1(S_1)\widehat{\otimes}\pi_2(S_2)(\sum_{i=1}^{n} h_{1_i} \otimes h_{2_i}) \\ &= \sum_{i=1}^{n} \pi_1(S_1)\widehat{\otimes}\pi_2(S_2)(h_{1_i} \otimes h_{2_i}) \\ &= \sum_{i=1}^{n} \pi_1(S_1)h_{1_i} \otimes \pi_2(S_2)h_{2_i} \\ &\supset \pi_1(S_1)h_{1_i} \otimes \pi_2(S_2)h_{2_i}\end{aligned}$$

for some $i$ $(1 \leq i \leq n)$. It follows that

$$\begin{aligned}\overline{\text{span } \pi_1 \otimes \pi_2(S_1 \times S_2)(v)} &\supset \overline{\text{span}\pi_1(S_1)h_{1_i} \otimes \pi_2(S_2)h_{2_i}} \\ &\supset \overline{\text{span } \pi_1(S_1)h_{1_i}} \otimes \overline{\text{span } \pi_2(S_2)h_{2_i}} \\ &= \mathcal{H}_1 \otimes \mathcal{H}_2.\end{aligned}$$

for some $i$ $(1 \leq i \leq n)$.

Thus $v$ is a cyclic vector for $\pi_1 \otimes \pi_2$ because $\mathcal{H}_1 \otimes \mathcal{H}_2$ is a dense subspace of $\mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2$. □

COROLLARY 10. *Let $S_1$ and $S_2$ be involutive semigroups, and let $\pi_1$ and $\pi_2$ be topologically irreducible representations of $S_1$ and $S_2$ in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively such that $\pi_1^{-1}(0) \neq \emptyset$ or $\pi_2^{-1}(0) \neq \emptyset$ and that either $\pi_1$ or $\pi_2$ is finite dimensional. Then either $\pi_1 \otimes \pi_2$ is a topologically irreducible representation of the product involutive semigroup $S_1 \times S_2$ in $\mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2$ or the case that dim $\pi_1 = 1$ and dim $\pi_2 > 1$ or dim $\pi_1 > 1$ and dim $\pi_2 = 1$.*

*Proof.* We note that if $\pi_1$ or $\pi_2$ is finite dimensional, $\mathcal{H}_1 \otimes \mathcal{H}_2 = \mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2$. If dim $\pi_1 = 1$ and dim $\pi_2 = 1$, then dim $\pi_1 \otimes \pi_2 = 1$ because dim $\mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2 = 1$ and hence $\pi_1 \otimes \pi_2$ is a topologically irreducible representation of $S_1 \times S_2$. In case that dim $\pi_1 > 1$ and dim $\pi_2 > 1$, by Theorem 9 and by Theorem 5, we can complete the proof. □

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Department of Mathematics, Kyungpook National University, Taegu 702-701, Korea