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THE AVERAGING THEOREM OF LEFSCHETZ COINCIDENCE NUMBERS AND ESTIMATION OF NIELSEN COINCIDENCE NUMBERS

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1. Coincidence Numbers

Throughout this paper, we will work in the category of compact connected manifolds and continuous maps. If X_1, X_2 are manifolds, f, g: $X_1 \to X_2$ are maps, then a point $x \in X_1$ is a coincidence of f and g iff $f(x) = g(x)$; the set of all such points is denoted by $Coin(f,g)$. If $x, y \in Coin(f,g)$, they are f,g-equivalence iff there exists a path ω in X from x to y with $f \circ \omega \simeq g \circ \omega$(rel 0,1). The f,g-equivalent is an eqivalence relation on $Coin(f,g)$ and a class by f,g-equivalent is called a coincidence class of f and g.

Suppose X_1, X_2 are both orientable n-manifolds. For each coincidence class C of f and g, a coincidence index $Ind(f,g,C)$is defined. If W, V are neighborhood of C with $C \subset W \subset \overline{W} \subset V$ and $Coin(f,g) \cap V = C$, then let $(f,g).$ be the composition

$$H_n(X_1) \longrightarrow H_n(X_1, X_1 \setminus W) \xleftarrow{\cong} H_n(V, V \setminus W) \xrightarrow{(f_*, g_*)} H_n(X_2 \times X_2, X_2 \times X_2 \setminus \Delta(X_2)).$$

If $z_1 \in H_n(X_1)$ is the fundamental class of X_1 and $U_2 \in H^n(X_2 \times X_2, X_2 \times X_2 \setminus \Delta(X_2))$ is the Thom class of X_2, then $Ind(f,g,C) =< U_2, (f,g).(z_1) >$. This is independent of V and W. and has the following properties :

(1) Coincidence : if $Ind(f,g,C) \neq 0$, then $C \neq \phi$

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(2) Homotopy : if $F\colon f_0 \simeq f_1$, $G\colon g_0 \simeq g_1$ and there exists a V such that $V \cap Coin(F,G)$ is compact, then $C_0 = V \cap Coin(f_0,g_0)$ and $C_1 = V \cap Coin(f_1,g_1)$ have $Ind(f_0,g_0,C_0) = Ind(f_1,g_1,C_1)$.

(3) Additivity : if $C = \cup C_i$, then $Ind(f,g,C) = \sum Ind(f,g,C_i)$.

(4) Products : Given $f,g\colon X_1 \to X_2$ and $f',g'\colon X_1', \to X_2'$ and $C \subseteq Coin(f,g)$, $C' \subseteq Coin(f',g')$, then $Ind(f,g,C)$ and $Ind(f',g',C')$ are defined iff $Ind(f \times f', g \times g', C \times C')$ is defined. If all are defined, $Ind(f \times f', g \times g', C \times C') = Ind(f,g,C) \cdot Ind(f',g',C')$.

A coincidence class C is essential if $Ind(f,g,C) \neq 0$. The Nielsen coincidence number $N(f,g)$ is the number of essential classes of f and g.

In rational coefficients, let $D_1\colon H^p(X_1) \to H_{n-p}(X_1)$ be the duality isomorphism and $\theta_p(f,g)$ be the composition

$$H_p(X_1) \xrightarrow{f_*} H_p(X_2) \xrightarrow{D_2^{-1}} H^{n-p}(X_2) \xrightarrow{g^*} H^{n-p}(X_1) \xrightarrow{D_1} H_p(X_1).$$

The Lefschetz coincidence number $L(f,g)$ is defined as $\Sigma_{p=0}^n(-1)^p tr(\theta_p(f,g))$, and the Lefschetz coincidence theorem states that $L(f,g) = Ind(f,g)$.

2. The Averaging Theorem of Lefschetz Coincidence Numbers and Estimation of Nielsen Coincidence numbers

Given $f,g\colon X_1 \to X_2$, fix base points $x_1 \in X_1$, $x_2 \in X_2$, and for convenience assume that $f(x_1) = x_2 = g(x_1)$. Let π_i denote $\pi(X_i,x_i)$ and define $\mathfrak{F}(\pi_i) = \{\Gamma \triangleleft \pi_i | [\pi_i\colon \Gamma] < \infty\}$. There is a one-to-one correspondence between elements of $\mathfrak{F}(\pi_i)$ and finite regular covers of X_i. If X_i is orientable, then the orientable finite regular covers of X_i corresponds to $\mathfrak{F}(\pi_i)$. Fix $\Gamma_2 \in \mathfrak{F}(\pi_2)$ and corresponding finite regular cover $p_2\colon \tilde{X}_2 \to X_2$. Given a cover $p_1\colon \tilde{X}_1 \to X_1$ and corresponding $\Gamma_1 \in \mathfrak{F}(\pi_1)$, f and g lift to some $\tilde{f},\tilde{g}\colon \tilde{X}_1 \to \tilde{X}_2$ iff $f_\sharp, g_\sharp\colon \pi_1 \to \pi_2$ have $f_\sharp, g_\sharp(\Gamma_1) \subseteq \Gamma_2$. So define $\mathfrak{F}(f,g,\Gamma_2) = \{\Gamma_1 \in \mathfrak{F}(\pi_1) | f_\sharp, g_\sharp(\Gamma_1) \subseteq \Gamma_2\}$.

We will refer to the lifting diagram

$$\begin{array}{ccc} \tilde{X}_1 & \xrightarrow{\tilde{f},\tilde{g}} & \tilde{X}_2 \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} \\ X_1 & \xrightarrow{f,g} & X_2 \end{array}$$

as the $\Gamma_1 - \Gamma_2$ lifting diagram of f and g. For any $\Gamma_1 - \Gamma_2$ lifting diagram, since the finite regular covers are compact, the lifts $\tilde{f}, \tilde{g}$ have the coincidence index defined.

In any $\Gamma_1 - \Gamma_2$ lifting diagram, Γ_i has covering group $\Phi_i = \pi_i/\Gamma_i$. $f_\sharp$ and $g_\sharp$ induce maps $\overline{f}, \overline{g}\colon \Phi_1 \to \Phi_2$. If $\tilde{f}, \tilde{g}\colon \tilde{X}_1 \to \tilde{X}_2$ are fixed "reference lifts" of f and g, then all lifts have the form $\beta \circ \tilde{f} \circ \alpha, \beta \circ \tilde{g} \circ \alpha$ with $\alpha \in \Phi_1, \beta \in \Phi_2$. But $\tilde{f} \circ \alpha = \overline{f}(\alpha) \circ \tilde{f}$, so we need only consider lifts of the form $\alpha \circ \tilde{f}, \alpha \circ \tilde{g}$.

PROPOSITION 2.1. *C is a coincidence class in $Coin(\alpha \circ \tilde{f}, \beta \circ \tilde{g})$ iff it is a coincidence class in $Coin(\beta^{-1}\alpha \circ \tilde{f}, \tilde{g})$.*

Thus, we need only consider lifting pairs of f, g of the form $(\alpha \circ \tilde{f}, \tilde{g})$.

We introduce the Reidemeister classes $\Phi_2/\sim$, where $\alpha \sim \beta$ iff $\overline{g}(\gamma)\beta = \alpha\overline{f}(\gamma)$ for some $\gamma \in \Phi_1$. The Reidemeister class of α is denoted by $[\alpha]$;

$$[\alpha] = \{\beta \in \Phi_2 | \overline{g}(\gamma)\beta = \alpha\overline{f}(\gamma) \text{ for some } \gamma \in \Phi_1\}.$$

The *mod*(Γ_1, Γ_2) Reidemeister number $R_{(\Gamma_1,\Gamma_2)}(f,g)$ is the number of Reidemeister classes.

PROPOSITION 2.2.

$$Cion(f,g) = \bigcup_{\alpha \in \Phi_2} p_1 Coin(\alpha \circ \tilde{f}, \tilde{g}) = \bigsqcup_{[\alpha]} p_1 Coin(\alpha \circ \tilde{f}, \tilde{g}).$$

Proof. By [3, Theorem 1.5], note that $p_1 Coin(\alpha \circ \tilde{f}, \tilde{g}) = p_1 Coin(\beta \circ \tilde{f}, \tilde{g})$ if $\beta \in [\alpha]$, and $p_1 Coin(\alpha \circ \tilde{f}, \tilde{g}) \cap p_1 Coin(\beta \circ \tilde{f}, \tilde{g}) = \phi$ if $\beta \notin [\alpha]$.

DEFINITION 2.3. *The number $\mu([\alpha])$ defined by $\mu([\alpha]) = |\{\gamma \in \Phi_1 | \overline{g}(\gamma)\alpha = \alpha\overline{f}(\gamma)\}|$ is called the multiplicity of class $[\alpha]$.*

It is easy to check that this number is independent of the choice of α in $[\alpha]$.

LEMMA 2.4. *If $x \in p_1 Coin(\alpha \circ \tilde{f}, \tilde{g})$, then $\mu([\alpha]) = |p_1^{-1}(x) \cap Coin(\alpha \circ \tilde{f}, \tilde{g})|$.*

Proof. Where $\tilde{x} \in p_1^{-1}(x) \cap Coin(\alpha \circ \tilde{f}, \tilde{g})$,it is easily shown that $\gamma\tilde{x} \in Coin(\alpha \circ \tilde{f}, \tilde{g})$ iff $\overline{g}(\gamma)\alpha = \alpha\overline{f}(\gamma)$.

PROPOSITION 2.5. *Suppose X_1, X_2 are compact connected orientable manifolds of the same dimensions and neither is a surface with negative Euler characteristic. If $\Gamma_1 \in \mathfrak{F}(f, g, \Gamma_2)$ and (f, g) lift to $(\tilde{f}, \tilde{g})$ in the $\Gamma_1 - \Gamma_2$ lifting diagram, then for any $\alpha \in \Phi_2$, $L(\alpha \circ \tilde{f}, \tilde{g}) = \mu([\alpha]) Ind(f, g, p_1 Coin(\alpha \circ \tilde{f}, \tilde{g}))$.*

Proof. We may assume that f and g have only isolated coincidences. Then we have $Ind(\alpha \circ \tilde{f}, \tilde{g}, \tilde{x}) = Ind(f, g, x)$ for $\tilde{x} \in p_1^{-1}(x)$. By Lemma 2.4, every coincidence of f and g in $p_1 Coin(\alpha \circ \tilde{f}, \tilde{g})$ has $\mu([\alpha])$ coincidences of $\alpha \circ \tilde{f}$ and $\tilde{g}$ above it. Hence

$$
\begin{aligned}
L(\alpha \circ \tilde{f}, \tilde{g}) &= \sum_{\tilde{x} \in Coin(\alpha \circ \tilde{f}, \tilde{g})} Ind(\alpha \circ \tilde{f}, \tilde{g}, \tilde{x}) \\
&= \mu([\alpha]) \sum_{x \in p_1 Coin(\alpha \circ \tilde{f}, \tilde{g})} Ind(f, g, x) \\
&= \mu([\alpha]) Ind(f, g, p_1 Coin(\alpha \circ \tilde{f}, \tilde{g})).
\end{aligned}
$$

LEMMA 2.6. $\mu([\alpha]) \cdot |[\alpha]| = |\Phi_1|$.

Proof. Φ_1 acts on $[\alpha]$ by $\alpha \xrightarrow{\gamma} \overline{g}(\gamma^{-1})\alpha\overline{f}(\gamma)$. The action is transitive, and the isotropy subgroup is $\{\gamma | \overline{g}(\gamma)\alpha = \alpha\overline{f}(\gamma)\}$.

THEOREM 2.7 (AVERAGING). *Suppose X_1, X_2 are compact connected orientable manifolds of the same dimensions, and neither is a* *surface with negative Euler characteristic. If $\Gamma_1 \in \mathfrak{F}(f,g,\Gamma_2)$ and (f,g) lift to $(\tilde{f},\tilde{g})$ in the $\Gamma_1 - \Gamma_2$ lifting diagram, then*

$$L(f,g) = \frac{1}{|\Phi_1|} \sum_{\alpha\in\Phi_2} L(\alpha \circ \tilde{f}, \tilde{g}).$$

Proof.

$$\begin{aligned}\sum_{\alpha\in\Phi_2} L(\alpha\circ\tilde{f},\tilde{g}) &= \sum_{\alpha\in\Phi_2} \mu([\alpha]) Ind(f,g,p_1Coin(\alpha\circ\tilde{f},\tilde{g}))\\ &= \sum_{[\alpha]} |[\alpha]|\mu([\alpha]) Ind(f,g,p_1Coin(\alpha\circ\tilde{f},\tilde{g}))\\ &= |\Phi_1|\sum_{[\alpha]} Ind(f,g,p_1Coin(\alpha\circ\tilde{f},\tilde{g}))\\ &= |\Phi_1|L(f,g).\end{aligned}$$

The following proposition corresponds to Theorem 4.15[3] by the same way theorem 3.2.8 [2] corresponds to Theorem 2.2.1 and 2.2.5 in fixed point theory.

PROPOSITION 2.8. *Suppose X_1, X_2 are compact connected manifolds. If $\Gamma_1 \in \mathfrak{F}(f,g,\Gamma_2)$, then $R_{(\Gamma_1,\Gamma_2)} \geq |Coker(g_{1*} - f_{1*})/\eta\cdot\theta(\Gamma_2)|$, where $\theta\colon \pi_1(Y) \to H_1(Y)$ is abelianization and $\eta\colon H_1(Y) \to Coker(H_1(Y) \xrightarrow{g_{1*}-f_{1*}} H_1(Y))$ is the natural projection. If Φ_2 is abelian, then equality holds.*

LEMMA 2.9. *Suppose X_1, X_2 are compact connected manifolds. If Φ_2 is abelian and $\Gamma_1 \in \mathfrak{F}(f,g,\Gamma_2)$, then $\mu([\alpha]) = \frac{|\Phi_1|}{|\Phi_2|}|Coker(g_{1*} - f_{1*})/\eta\cdot\theta(\Gamma_2)|$ for all $\alpha \in \Phi_2$.*

Proof. By Proposition 2.8, we have $R_{(\Gamma_1,\Gamma_2)}(f,g) = |Coker(g_{1*} - f_{1*})/\eta\cdot\theta(\Gamma_2)|$. Since Φ_2 is abelian, there is one-to-one corresponding from $[\alpha]$ to $[\beta]$ defined by

$$\gamma \longrightarrow \gamma\alpha^{-1}\beta$$

for any $\gamma \in [\alpha]$. So all Reidemeister classes of f and g consist of the same number(say k), and $k \cdot |Coker(g_{1*} - f_{1*})/\eta \cdot \theta(\Gamma_2)| = |\Phi_2|$. Lemma 2.6 tell us $k \cdot \mu([\alpha]) = |\Phi_1|$. Hence

$$\mu([\alpha]) = \frac{|\Phi_1|}{|\Phi_2|}|Coker(g_{1*} - f_{1*})/\eta \cdot \theta(\Gamma_2)|.$$

Now we consider the case that π_1 and π_2 are finite groups. Then we can apply trivial subgroups of π_1 and π_2 to above results. In this case, the $mod(\Gamma_1, \Gamma_2)$ Reidemeister number is the Reidemeister number $R(f,g)$ of f and g and a nonempty set $p_1 Coin(\alpha \circ \tilde{f}, \tilde{g})$ equals to a coincidence class of f and g.

The argument in Theorem 2.5.6[2] also establishes the following, quoting Proposition 2.5 and Lemma 2.9.

THEOREM 2.10. *Suppose X_1, X_2 are compact connected orientable manifolds of the same dimensions with the finite fundamental groups π_1 and π_2, and neither is a surface with negative Euler characteristic. Assume the action of π_2 on the rational homology of the universal covering space $\tilde{X}_2$ of X_2 is trivial, i.e., for every covering translation $\alpha \in \pi_2$, $\alpha_* = id$: $H_*(\tilde{X}_2 : Q) \to H_*(\tilde{X}_2 : Q)$. Then, for every $f, g: X_1 \to X_2$, $L(f,g) = 0$ implies $N(f,g) = 0$; $L(f,g) \neq 0$ implies $N(f,g) = R(f,g) \geq |Coker(g_{1*} - f_{1*})|$. If π_2 is abelian, then $R(f,g) = |Coker(g_{1*} - f_{1*})|$ and the coincidence index of the coincidence classes are all equal.*

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