Bull. Korean Math. Soc. **33** (1996), No. 2, pp. 193–204

BEST SIMULTANEOUS APPROXIMATIONS FROM A CONVEX SUBSET

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1. Introduction

Let U and V be nonempty compact subsets of two Hausdorff topological vector spaces. Suppose that a function $J : U \times V \to \mathbb{R}$ is such that for each $v \in V$, $J(\cdot, v)$ is lower semi-continuous and convex on U, and for each $u \in U$, $J(u, \cdot)$ is upper semi-continuous and concave on V. Then, as is well known [2], there exists a saddle point $(u^*, v^*) \in U \times V$ such that

$$J(u^*, v) \leq J(u^*, v^*) \leq J(u, v^*), \quad u \in U,\ v \in V, \tag{1.1}$$

that is,

$$\min_{u \in U} \max_{v \in V} J(u, v) = \max_{v \in V} \min_{u \in U} J(u, v).$$

However, if V is not convex, or if for some $u \in U$, $J(u, \cdot)$ is not a concave function on V, then the relation (1.1) does not hold in general.

THEOREM 1.1.[4]. *Let U be an n-dimensional, compact convex set in a Hausdorff topological vector space ($n \geq 1$) and let V be a compact Hausdorff space. Let $J : U \times V \to \mathbb{R}$ be a jointly continuous function. Then $u^* \in U$ minimizes $\max_{v \in V} J(u, v)$ over U if and only if there exists $(\overline{\lambda}^*_{n+1}, \overline{v}^*_{n+1}) \in \overline{V}_{n+1}$ such that*

$$\sum_{i=1}^{n+1} \lambda_i J(u^*, v_i) \leq \sum_{i=1}^{n+1} \lambda_i^* J(u^*, v_i^*) \leq \sum_{i=1}^{n+1} \lambda_i^* J(u, v_i^*) \tag{1.2}$$

Received October 4, 1994.

1991 AMS Subject Classification: 41A28.

Key words and phrases: Best uniform approximation, Best simultaneous approximation.

*Supported partially by the Basic Science Research Institute program, Ministry of Education.

holds for all $(\overline{\lambda}_{n+1}, \overline{v}_{n+1}) \in \overline{V}_{n+1}$ *and for all* $u \in U$*, where*

$$\overline{V}_{n+1} = \{(\overline{\lambda}_n, \overline{v}_n) | \overline{\lambda}_n = (\lambda_1, \cdots, \lambda_n), \overline{v} = (v_1, \cdots, v_n), \sum_{i=1}^{n} \lambda_i = 1, \lambda_i \geq 0, v_i \in V (i = 1, \cdots, n)\}.$$

2. Best Uniform Approximations

Let X be a compact Hausdorff space and let Y be a normed linear space with a norm $||\cdot||$. Let $C(X, Y)$ denote the set of all continuous functions from X to Y. Let C be a convex set in an n-dimensional subspace of $C(X, Y)$ and $F \in C(X, Y)$. An element $f^* \in C$ is called a best uniform approximation to F from C if f^* minimizes

$$\max_{x \in X} ||f(x) - F(x)||$$

over C, that is, if

$$\max_{x \in X} ||f^*(x) - F(x)|| \leq \max_{x \in X} ||f(x) - F(x)||$$

holds for all $f \in C$. For $f \in C(X, Y)$, we define the uniform norm of f by

$$|||f||| = \max_{x \in X} ||f(x)||,$$

and endow the linear space $C(X, Y)$ with the uniform topology. Clearly $||f(x) - F(x)||$ is a jointly continuous function of two variables f, x and convex in f, i.e.,

$$||(\lambda f + (1 - \lambda)g)(x) - F(x)|| \leq \lambda ||f(x) - F(x)|| + (1 - \lambda)||g(x) - F(x)||$$

for all f, $g \in C(X, Y)$, $x \in X$, and λ, $0 \leq \lambda \leq 1$. Hence if C is a convex set in a finite dimensional subspace of $C(X, Y)$, we can apply Theorem 1.1 and obtain the following necessary and sufficient condition of a best uniform approximation from C.

THEOREM 2.1. *Let C be a closed convex set in an n-dimensional subspace of $C(X,Y)$ and $F \in C(X,Y)$. Then $f^* \in C$ is a best uniform approximation to F from C if and only if there exist $\lambda_1^*, \cdots, \lambda_k^* > 0$, $\sum_{i=1}^k \lambda_i^* = 1$, and k distinct elements $x_1^*, \cdots, x_k^*$ of X, where $1 \leq k \leq n+1$, satisfying*

$$\text{(i)} \qquad \|f^*(x_i^*) - F(x_i^*)\| = |||f^* - F||| \quad i = 1, 2, \cdots, k;$$

$$\text{(ii)} \qquad \sum_{i=1}^{k} \lambda_i^* \|f(x_i^*) - F(x_i^*)\| \geq \sum_{i=1}^{k} \lambda_i^* \|f^*(x_i^*) - F(x_i^*)\|$$

for any $f \in C$.

Proof. ($\Rightarrow$) Let f^* be a best uniform approximation to F from C and let $U = \{f \in C \mid \|f^* - f\| \leq 1\}$. Then f^* minimizes $\max_{x \in X} \|f(x) - F(x)\|$ over U and U is a compact subset of C because it is bounded and closed in an n-dimensional subspace. By Theorem 1.1, there exist $\lambda_1', \cdots, \lambda_{n+1}' \geq 0$, $\sum_{i=1}^{n+1} \lambda_i' = 1$, and $\{x_1', \cdots, x_{n+1}'\} \subset X$ such that

$$(2.1) \qquad \sum_{i=1}^{n+1} \lambda_i' \|f^*(x_i') - F(x_i')\| \geq \sum_{i=1}^{n+1} \lambda_i \|f^*(x_i) - F(x_i)\|$$

for all $(\overline{\lambda}_{n+1}, \overline{x}_{n+1}) \in \overline{X}_{n+1} = \{(\overline{\lambda}_{n+1}, \overline{x}_{n+1}) : \overline{\lambda}_{n+1} = (\lambda_1, \cdots, \lambda_{n+1}),$

$$\overline{x}_{n+1} = (x_1, \cdots, x_{n+1}), \sum_{i=1}^{n+1} \lambda_i = 1, \lambda_i \geq 0, x_i \in X (i = 1, \cdots, n+1)\};$$

$$(2.2) \qquad \sum_{i=1}^{n+1} \lambda_i' \|f(x_i') - F(x_i')\| \geq \sum_{i=1}^{n+1} \lambda_i' \|f^*(x_i') - F(x_i')\|$$

for all $f \in U$. Let us denote the nonzero elements within $\lambda_1', \cdots, \lambda_{n+1}'$ by $\lambda_1^*, \cdots, \lambda_k^*$ and the corresponding elements within $x_1', \cdots, x_{n+1}'$ of X by $x_1^*, \cdots, x_k^*$. The assertion (i) follows from (2.1) which means, for $i = 1, \cdots, k$,

$$\|f^*(x_i^*) - F(x_i^*)\| = \max_{x \in X} \|f^*(x) - F(x)\| = |||f^* - F|||.$$

On the other hand, it follows from (2.2) that

$$\sum_{i=1}^{k} \lambda_i^* ||f(x_i^*) - F(x_i^*)|| \geq \sum_{i=1}^{k} \lambda_i^* ||f^*(x_i^*) - F(x_i^*)||$$

holds for all $f \in U$. Since the left-hand side of the above inequality is a convex function of f and has a local minimum at f^*, f^* realizes a global minimum by a property of convex functions. Thus ii) holds.

($\Leftarrow$) Conversely, suppose that (i) and (ii) hold. These two conditions yield

$$\begin{aligned}\sup_{(\overline{\lambda}_k, \overline{x}_k) \in \overline{X}_k} \inf_{f \in C} \sum_{i=1}^{k} \lambda_i ||f(x_i) - F(x_i)|| &\geq \sum_{i=1}^{k} \lambda_i^* ||f^*(x_i^*) - F(x_i^*)|| \\ &= |||f^* - F|||,\end{aligned}$$

where $\overline{X}_k$ is similarly defined as the set $\overline{X}_{n+1}$. The left-hand side of the last relation is equal to or less than

$$\inf_{f \in C} \max_{(\overline{\lambda}_k, \overline{x}_k) \in \overline{X}_k} \sum_{i=1}^{k} \lambda_i ||f(x_i) - F(x_i)|| = \inf_{f \in C} \max_{x \in X} ||f(x) - F(x)||,$$

so

$$|||f^* - F||| \leq \inf_{f \in C} |||f - F|||.$$

Therefore, f^* is a best uniform approximation to F from C.

REMARK. The proof of Theorem 2.1 is the same as that of Theorem 4.1 in [4].

The next corollary states that f^* is a best uniform approximation on X if and only if it also is on some finite subset of X.

COROLLARY 2.2. *Let C be a closed convex set in an n-dimensional subspace of $C(X, Y)$ and $F \in C(X, Y)$. Then f^* is a best uniform approximation to F from C if and only if there exist k elements $x_1^*, \cdots, x_k^* \in X$, where $1 \leq k \leq n+1$, for which f^* satisfies*

(i) $$||f^*(x_j^*) - F(x_j^*)|| = |||f^* - F|||, \quad j = 1, \cdots, k;$$

(ii) $$\max_{1 \leq j \leq k} ||f(x_j^*) - F(x_j^*)|| \geq \max_{1 \leq j \leq k} ||f^*(x_j^*) - F(x_j^*)||$$

for all $f \in C$.

COROLLARY 2.3 [3]. *Let C be an n-dimensional subspace of $C(X,Y)$ and $F \in C(X,Y)$. Then $f^* \in C$ is a best uniform approximation to F from C if and only if there exist $\lambda_1^*, \cdots, \lambda_k^* > 0$, $\sum_{i=1}^k \lambda_i^* = 1$, and k distinct elements $x_1^*, \cdots, x_k^*$ of X, where $1 \leq k \leq n+1$, satisfying*

(i) $$||f^*(x_i^*) - F(x_i^*)|| = |||f^* - F|||,\ i = 1, \cdots k;$$

(ii) $$\sum_{i=1}^k \lambda_i^* f(x_i^*) - F(x_i^*)|| \geq \sum_{i=1}^k \lambda_i^* ||f^*(x_i^*) - F(x_i^*)||$$

for all $f \in C$.

Proof. The conclusion holds since an n-dimensional subspace is a closed convex set.

3. Best Simultaneous Approximations

Suppose that functions $F_1, \cdots, F_\ell$ are in $C(X,Y)$. Define a norm on the linear space of ℓ-tuples of elements of $C(X,Y)$ as follows: for any $F_1, \cdots, F_\ell$ in $C(X,Y)$ let $\mathbf{F} = (F_1, \cdots, F_\ell)$ and

$$|||\mathbf{F}||| = |||(F_1, \cdots, F_\ell)||| = \max_{||\mathbf{a}||_1} |||\sum_{i=1}^{\ell} a_i F_i|||.$$

We want to approximate these functions simultaneously by functions in a closed convex set C in an n-dimensional subspace of $C(X,Y)$. That is, the problem is to find a function $f \in C$ which minimizes

$$\max_{1 \leq j \leq \ell} \max_{x \in X} ||F_j(x) - f(x)|| \tag{3.1}$$

over C. If such a function f^* in C exists, we call it a best simultaneous approximation for $\mathbf{F} = (F_1, \cdots, F_\ell)$ from C.

In this section, we want to derive a necessary and sufficient condition for a function to be a best simultaneous approximation. It is easy to show that (3.1) can be expressed as

$$\max_{\mathbf{a} \in A} \max_{x \in X} ||\sum_{j=1}^{\ell} \alpha_j F_j(x) - f(x)|| = \max_{\mathbf{a} \in A} \max_{x \in X} ||\mathbf{aF}(x) - f(x)||$$

where the set A is defined by

$$A = \{\mathbf{a} = (\alpha_1, \cdots, \alpha_\ell) : \sum_{j=1}^{\ell} \alpha_j = 1,\ \alpha_j \geq 0 (1 \leq j \leq \ell)\}.$$

If $\sum_{j=1}^{\ell} \alpha_j F_j(x)$ is denoted by the inner product $\mathbf{a}\mathbf{F}(x)$ of two vectors $\mathbf{a} = (\alpha_1, \alpha_2, \cdots, \alpha_\ell)$ and $\mathbf{F}(x) = (F_1(x), \cdots, F_\ell(x))$, then the problem takes on the expression

$$(3.2) \qquad \text{minimize} \max_{(\mathbf{a},x)\in A\times X} ||\mathbf{a}\mathbf{F}(x) - f(x)|| \text{ over the set } C.$$

Note that $A \times X$ is compact, and $||\mathbf{a}F(x) - f(x)||$ is a jointly continuous function of three variables $\mathbf{a}$, x, f and convex in f, that is,

$$||\mathbf{a}\mathbf{F}(x) - (\theta f + (1-\theta)g)(x)|| \leq \theta ||\mathbf{a}\mathbf{F}(x) - f(x)|| + (1-\theta)||\mathbf{a}\mathbf{F}(x) - g(x)||$$

for all f, $g \in C$ and θ, $0 \leq \theta \leq 1$.

THEOREM 3.1. *Let C be a closed convex set in an n-dimensional subspace of $C(X, Y)$ and let $\mathbf{F} = (F_1, \cdots, F_\ell)$. Then $f^* \in C$ is a best simultaneous approximation for $\mathbf{F}$ from C if and only if there exist $\lambda_1^*, \cdots, \lambda_k^* > 0$, $\sum_{i=1}^{k} \lambda_i^* = 1$, k distinct elements $x_1^*, \cdots, x_k^* \in X$, and k vectors $\mathbf{a}_1^*, \cdots, \mathbf{a}_k^* \in A$, where $1 \leq k \leq n+1$, such that*

(i)
$$||\mathbf{a}_i^*\mathbf{F}(x_i^*) - f^*(x_i^*)|| = \max_{1\leq j\leq \ell} ||F_j(x_i^*) - f^*(x_i^*)|| = \max_{1\leq j\leq \ell} |||F_j - f^*||| \quad i = 1, \cdots, k;$$

(ii)
$$\sum_{i=1}^{k} \lambda_i^* ||\mathbf{a}_i^*\mathbf{F}(x_i^*) - f(x_i^*)|| \geq \sum_{i=1}^{k} \lambda_i^* ||\mathbf{a}_i^*\mathbf{F}(x_i^*) - f^*(x_i^*)||$$

for any $f \in C$.

Proof. By the preceding remarks concerning the function $||\mathbf{a}F(x) - f(x)||$, we can apply Theorem 1.1. The rest of the proof is similar to that of Theorem 2.1.

Recall that the subdifferential or set of subgradients of $|||\cdot|||$ at any element f of $C(X,Y)$ is the set defined by

$$\partial|||f||| = \{w \in C^*(X,Y) : |||g||| \geq |||f||| \\ + < w, g-f > \text{ for any } g \in C(X,Y)\}$$

where the usual inner product notation is used to link elements of $C(X,Y)$ and its dual $C^*(X,Y)$. For an ℓ-tuple $\Phi = (\phi_1, \cdots, \phi_\ell)$ of functions in $C(X,Y)$, define the set

$$G(\Phi) = \{(\mathbf{a}, w) : \mathbf{a} \in \mathbb{R}^\ell, ||\mathbf{a}||_1 = 1, \\ \sum_{i=1}^{\ell} a_i\phi_i = |||\Phi|||u, |||u||| = 1, w \in \partial|||u|||\}.$$

Now let

$$\phi_i(f) = F_i - f, \quad i = 1, \cdots, \ell$$

for all $f \in C$ and let $H(f)$ denote the set of ℓ-tuples $\{h_1, \cdots, h_\ell\}$ of elements in $C^*(X,Y)$ defined by

$$H(f) = conv\{(a_1\omega, \cdots, a_\ell\omega) : (\mathbf{a}, \omega) \in G(\Phi(f))\},$$

where as usual '$conv$' is used to denote the convex hull. Note that

$$\sum_{i=1}^{\ell} < h_i, \phi_i(f) >= ||\Phi(f)|| \; for\; all\; \mathbf{h} \in H(f).$$

For $f \in C$ and any $\mathbf{a} \in \mathbb{R}^\ell$ with $||\mathbf{a}||_1 = 1$, define

$$E(\mathbf{a}, f) = \{x \in X : ||\sum_{i=1}^{\ell} a_i\phi_i(f(x))||_Y = |||\Phi(f)|||\},$$

and let

$$E(f) = \{x \in X : ||\sum_{i=1}^{\ell} a_i\phi_i(f(x))||_Y = |||\Phi(f)||| \text{ for some } \mathbf{a}, ||\mathbf{a}|| = 1\}.$$

Then for each distinct $x_k \in E(f)$, there exists $\mathbf{a}_k$, $||\mathbf{a}_k||_1 = 1$, such that

$$x_k \in E(\mathbf{a}_k, f). \tag{3.3}$$

In particular, $(\mathbf{a}, w) \in G(\Phi(f))$ implies that

$$w \in conv\{v(x)\delta(x) : x \in E(\mathbf{a}, f)\},$$

where δ is the delta function, and $v(x) \in \partial||\sum_{i=1}^{\ell} \alpha_i \phi_i(f(x))||_Y$. Thus

$$< w, f >= \sum_{k=1}^{s} \mu_k < v(x_k), f(x_k) >_Y,$$

where $< \cdot, \cdot >_Y$ denotes an inner product between elements of Y and its dual, where $x_k \in E(\mathbf{a}, f)$ and μ_k, $k = 1, \cdots, s$, are nonnegative numbers summing to 1.[See [5]]

COROLLARY 3.2. *Let C be a closed convex set in an n-dimensional subspace of $C(X, Y)$. Then $f^* \in C$ is a best simultaneous approximation for $\mathbf{F}$ from C if and only if there exist $m(\leq n+1)$ distinct elements $x_1, \cdots, x_m$ of $E(f^*)$, m vectors $\mathbf{a}_1, \cdots, \mathbf{a}_m$ satisfying $x_k \in E(\mathbf{a}_k, f^*)(k = 1, \cdots, m)$ and m positive numbers $\gamma_1, \cdots, \gamma_m$ summing to 1 such that*

$$\sum_{k=1}^{m} \gamma_k ||\mathbf{a}_k \mathbf{F}(x_k) - f(x_k)|| \geq \sum_{k=1}^{m} \gamma_k ||\mathbf{a}_k \mathbf{F}(x_k) - f^*(x_k)||$$

for any $f \in C$.

Proof. By Theorem 3.1 and the above argument, f^* is a best simultaneous approximation for $\mathbf{F}$ from C if and only if there exist $\gamma_1, \cdots, \gamma_m > 0$, $\sum_{k=1}^{m} \gamma_k = 1$, m distinct elements $x_1, \cdots, x_m \in X$ and k vectors $\mathbf{a}_1, \cdots, \mathbf{a}_m \in A$, where $1 \leq k \leq n+1$, such that

(1)
$$\begin{aligned} ||\mathbf{a}_k \mathbf{F}(x_k) - f^*(x_k)|| &= \max_{1 \leq j \leq \ell} ||F_j(x_k) - f^*(x_k)|| \\ &= \max_{1 \leq j \leq \ell} |||F_j - f^*||| \quad k = 1, \cdots, m \end{aligned}$$

(2)

$$\sum_{k=1}^{m} \gamma_k ||\mathbf{a}_k \mathbf{F}(x_k) - f(x_k)|| \geq \sum_{k=1}^{k} \gamma_k ||\mathbf{a}_k \mathbf{F}(x_k) - f^*(x_k)||$$

for any $f \in C$ if and only if there exist $m(\leq n+1)$ distinct elements $x_1, \cdots, x_m$ of $E(f^*)$, m vectors $\mathbf{a}_1, \cdots, \mathbf{a}_m$ satisfying $x_k \in E(\mathbf{a}_k, f^*)(k = 1, \cdots, m)$ and m positive numbers $\gamma_1, \cdots, \gamma_k$ summing to 1 such that

$$\sum_{k=1}^{m} \gamma_k ||\mathbf{a}_k \mathbf{F}(x_k) - f(x_k)|| \geq \sum_{k=1}^{m} \gamma_k ||\mathbf{a}_k \mathbf{F}(x_k) - f^*(x_k)||$$

for any $f \in C$.

Since an n-dimensional subspace is a closed convex set, we obtain the next corollary.

COROLLARY 3.3. [4]. *Let C be an n-dimensional subspace of $C(X, Y)$. Then $f^* \in C$ is a best simultaneous approximation for $\mathbf{F}$ from C if and only if there exist $\lambda_1^*, \cdots, \lambda_k^* > 0$, $\sum_{i=1}^{k} \lambda_i^* = 1$, k distinct elements $x_1^*, \cdots, x_k^* \in X$, and k vectors $\mathbf{a}_1^*, \cdots, \mathbf{a}_k^* \in A$, where $1 \leq k \leq n+1$, such that*

$$\text{(i)} \qquad ||\mathbf{a}_i^* \mathbf{F}(x_i^*) - f^*(x_i^*)|| = \max_{1 \leq j \leq \ell} ||F_i(x_i^*) - f^*(x_i^*)|| = \max_{1 \leq j \leq \ell} |||F_j - f^*||| \quad i = 1, \cdots, k;$$

(ii)

$$\sum_{i=1}^{k} \lambda_i^* ||\mathbf{a}_i^* \mathbf{F}(x_i^*) - f(x_i^*)|| \geq \sum_{i=1}^{k} \lambda_i^* ||\mathbf{a}_i^* \mathbf{F}(x_i^*) - f^*(x_i^*)||$$

for any $f \in C$.

COROLLARY 3.4. *Let C be a closed convex subset in an n-dimensional subspace of $C(X, Y)$ and let $\mathbf{F} = (F_1, \cdots, F_\ell)$. If there exist $m(\leq n+1)$ distinct elements $x_1, \cdots, x_m$ of $E(f^*)$, m vectors $\mathbf{a}_1, \cdots, \mathbf{a}_m$ satisfying with $f = f^*$, and m positive numbers $\gamma_1, \cdots, \gamma_m$ summing to 1 such that*

$$\text{(i)} \qquad \sum_{k=1}^{m} \gamma_k < v_k(x_k), f^*(x_k) >_Y = 0;$$

$$\text{(ii)} \qquad \sum_{k=1}^{m} \gamma_k < v_k(x_k), f(x_k) >_Y \leq 0$$

for any $f \in C$, *where* $v_k(x) \in \partial || \sum_{i=1}^{\ell} a_k^i \phi_i(f^*(x))||_Y, k = 1, 2, \cdots, m$, *then* $f^* \in C$ *is a best simultaneous aproximation for* $\mathbf{F}$ *from* C.

Proof. By Theorem 3.1, it sufices to show that

$$\sum_{k=1}^{m} \gamma_k || \sum_{i=1}^{\ell} a_k^i \phi_i(f^*(x_k))||_Y \leq \sum_{k=1}^{m} \gamma_k || \sum_{i=1}^{\ell} a_k^i \phi_i(f(x_k))||^Y$$

for any $f \in C$. Let $f \in C$ be arbitrary, and let

$$v_k(x) \in \partial || \sum_{i=1}^{\ell} a_k^i \phi_i(f^*(x_k))||_Y \quad k = 1, 2, \cdots, m$$

satisfy (i) and (ii). Then

$$\begin{aligned} &\sum_{k=1}^{m} \gamma_k || \sum_{i=1}^{\ell} a_k^i \phi_i(f^*(x_k))||_Y \\ &= \sum_{k=1}^{m} \gamma_k < v_k(x_k), \sum_{i=1}^{\ell} a_i^k (F_i(x_k) - f^*(x_k)) >_Y \\ &\leq \sum_{k=1}^{k} \gamma_k < v_k(x_k), \sum_{i=1}^{\ell} a_i^k (F_i(x_k) - f(x_k)) >_Y \\ &\leq \sum_{k=1}^{m} \gamma_k || \sum_{i=1}^{\ell} a_k^i \phi_i(f(x_k))||_Y. \end{aligned}$$

Thus f^* is a best simultaneous approximation for $\mathbf{F}$ from C.

COROLLARY 3.5. *Let* C *be a convex subset in an n-dimensional subspace of* $C(X, Y)$ *and let* $\mathbf{F} = (F_1, \cdots, F_\ell)$. *If there exists* $\mathbf{h} = (h_1, h_2, \cdots, h_\ell) \in H(f^*)$ *such that*

(i) $$\sum_{i=1}^{\ell} < h_i, f - f^* > \leq 0$$

for all $f \in C$, then $f^* \in C$ is a best simultaneous approximation for $\mathbf{F}$ *from C.*

Proof. Let f be any element of C. Then

$$\begin{aligned} ||\phi(f)|| &= \max_{||\mathbf{a}||_1=1} ||\sum_{i=1}^{\ell} a_i(F_i - f)||_A \\ &\geq < w, \sum_{i=1}^{\ell} a_i(F_i - f) > \end{aligned}$$

for all $(\mathbf{a}, w) \in G(\Phi(f^*))$. Suppose that $\mathbf{h} \in H(f^*)$ satisfies (i). Then

$$\begin{aligned} ||\phi(f)|| &\geq \sum_{i=1}^{\ell} < h_i, F_i - f > \\ &= \sum_{i=1}^{\ell} < h_i, F_i - f^* + f^* - f > \\ &\geq \sum_{i=1}^{\ell} < h_i, F_i - f > \\ &= ||\phi(f^*)||. \end{aligned}$$

Thus f^* is a best simultaneous approximation for $\mathbf{F}$ from C.

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