Bull. Korean Math. Soc. **33** (1996), No. 1, pp. 135–170

SOBOLEV ORTHOGONAL POLYNOMIALS AND SECOND ORDER DIFFERENTIAL EQUATION II

K. H. KWON, D. W. LEE, AND L. L. LITTLEJOHN

1. Introduction

Recently many people have studied the Sobolev orthogonal polynomials, that is, polynomials which are orthogonal relative to a symmetric bilinear form $\phi(\cdot,\cdot)$ defined by

$$\phi(p,q) := (p,q)_N = \sum_{k=0}^{N} \int_{\mathbb{R}} p^{(k)}(x) q^{(k)}(x) d\mu_k, \tag{1.1}$$

where each $d\mu_k$ is a signed Borel measure on the real line $\mathbb{R}$ with finite moments of all orders. For the brief history on this subject, we refer to the survey article Ronveaux [13] and Marcellán and et al [10].

For $N = 0$, Bochner [1] classified orthogonal polynomials relative to $(\cdot,\cdot)_0$ satisfying a second order differential equation of the form

$$\ell_2(x)y''(x) + \ell_1(x)y'(x) = \lambda_n y(x), \tag{1.2}$$

which are now called the classical orthogonal polynomials (see also [7]).

For $N = 1$, Kwon and Littlejohn [8,9] found necessary and sufficient conditions on coefficients $\ell_2(x)$ and $\ell_1(x)$ for a sequence of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ to be orthogonal relative to Sobolev inner-product $(\cdot,\cdot)_1$ and to satisfy a second order differential equation of the form (1.2) and then they showed that, up to a real linear change of variable, there are eleven such distinct Sobolev orthogonal polynomials including six classical orthogonal polynomials.

Received September 27, 1994.

1991 AMS Subject Classification: 33C45.

Key words and phrases: Sobolev orthogonal polynomials, second-order differential equations.

In this work, we find necessary and sufficient conditions for a sequence of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ satisfying the differential equation (1.2) to be orthogonal relative to Sobolev inner product $(\cdot,\cdot)_N$, $N \geq 0$, and then classify all such Sobolev orthogonal polynomials when $N = 2$.

2. Preliminaries

All polynomials throughout this work are assumed to be real polynomials of a real variable x ; the space of all such polynomials is denoted by $\mathcal{P}$. We shall denote the degree of a polynomial $\pi \in \mathcal{P}$ by $\deg(\pi)$, with the convention that $\deg(0) = -1$. By a polynomial system (PS), we mean a sequenceof polynomials $\{\phi_n(x)\}_{n=0}^{\infty}$ with $\deg(\phi_n) = n$ $(n \geq 0)$; in this case $\{\phi_n(x)\}_{n=0}^{\infty}$ form a basis for $\mathcal{P}$. We call any linear functional $\sigma : \mathcal{P} \to \mathbb{R}$ a moment functional and denote its action on a polynomial π by $\langle \sigma, \pi \rangle$. With this action, any moment functional σ defines a symmetric bilinear formon $\mathcal{P} \times \mathcal{P}$ by the formula $\langle \sigma, pq \rangle$ $(p, q \in \mathcal{P})$.

We say that a moment functional σ is quasi-definite (respectively, positive-definite) if the moments

$$\sigma_n := \langle \sigma, x^n \rangle \qquad (n \geq 0)$$

of σ satisfy the Hamburger condition

$$\Delta_n(\sigma) := \det[\sigma_{i+j}]_{i,j=0}^{n} \neq 0 \quad \text{(respectively, } \Delta_n(\sigma) > 0) \tag{2.1}$$

for each $n \geq 0$.

More generally for any symmetric bilinear form $\phi(\cdot,\cdot)$ on $\mathcal{P} \times \mathcal{P}$, we call the double sequence

$$\phi_{m,n} := \phi(x^m, x^n) \qquad (m \text{ and } n \geq 0)$$

the moments of $\phi(\cdot,\cdot)$ and say that $\phi(\cdot,\cdot)$ isquasi-definite (respectively, positive-definite) if

$$\Delta_n(\phi) := \det[\phi_{i,j}]_{i,j=0}^{n} \neq 0 \quad \text{(respectively, } \Delta_n(\phi) > 0) \tag{2.2}$$

for each $n \geq 0$.

LEMMA 2.1. *A symmetric bilinear form* $\phi(\cdot,\cdot)$ *on* $\mathcal{P}\times\mathcal{P}$ *is quasi-definite (respectively, positive-definite) if and only if there are PS* $\{P_n(x)\}_{n=0}^{\infty}$ *and real constants* $K_n \neq 0$ *(respectively,* $K_n > 0$*) for* $n \geq 0$ *such that*

$$\phi(P_m, P_n) = K_n\delta_{mn} \qquad (m \text{ and } n \geq 0). \tag{2.3}$$

Moreover, in this case, each $P_n(x)$ *is uniquely determined up to a nonzero constant multiple.*

Proof. Assume that $\phi(\cdot,\cdot)$ is quasi-definite. Define a sequence of polynomials by

$$P_0(x) := 1$$

and
(2.4)

$$P_n(x) := [\Delta_{n-1}(\phi)]^{-1}\det\begin{bmatrix}\phi_{0,0} & \phi_{0,1} & \cdots & \phi_{0,n}\\ \phi_{1,0} & \phi_{1,1} & \cdots & \phi_{1,n}\\ \vdots & \vdots & \ddots & \vdots\\ \phi_{n-1,0} & \phi_{n-1,1} & \cdots & \phi_{n-1,n}\\ 1 & x & \cdots & x^n\end{bmatrix} \qquad (n \geq 1).$$

Then $\{P_n(x)\}_{n=1}^{\infty}$ is a monic PS and we have (2.3) with

$$K_n = \Delta_n(\phi)/\Delta_{n-1}(\phi) \qquad (n \geq 0),$$

where $\Delta_{-1}(\phi) = 1$. Now let $\{\widetilde{P}_n(x)\}_{n=0}^{\infty}$ be another PS satisfying

$$\phi(\widetilde{P}_m, \widetilde{P}_n) = \widetilde{K}_n\delta_{mn} \qquad (m \text{ and } n \geq 0), \tag{2.5}$$

where $\widetilde{K}_n \neq 0$. Since $\{P_n(x)\}_{n=0}^{\infty}$ is a PS, for each $n \geq 0$ we may write $\widetilde{P}_n(x)$ as

$$\widetilde{P}_n(x) = \sum_{j=0}^{n} C_{n,j}P_j(x)$$

for some real constants $C_{n,j}$ $(j = 0, 1, \ldots, n)$ with $C_{n,n} \neq 0$. Then, for any $k = 0, 1, \ldots, n-1$, we have, by the orthogonality (2.3),

$$0 = \phi(\widetilde{P}_n, P_k) = \sum_{j=0}^{n} C_{n,j}\phi(P_j, P_k) = \sum_{j=0}^{N} C_{n,j}K_j\delta_{jk} = C_{n,k}K_k$$

so that $C_{n,k} = 0$. Hence we have $\widetilde{P}_n(x) = C_{n,n}P_n(x)$. Conversely, assume that there is a PS $\{P_n(x)\}_{n=0}^{\infty}$ satisfying (2.3). We may assume that each $P_n(x)$ is monic so that $\{P_n(x)\}_{n=0}^{\infty}$ is the unique monic PS satisfying (2.3). Writing $P_n(x) = \sum_{k=0}^{n} C_k^n x^k$ ($C_n^n = 1$), we see that the condition (2.3) isequivalent to

$$(2.6) \qquad \sum_{k=0}^{n} \phi_{m,k} C_k^n = \phi(x^m, P_n(x)) = \phi(P_m(x), P_n(x)) = K_n \delta_{mn}$$

for $n \geq 0$ and $m = 0, 1, \ldots, n$. Since for any fixed $n \geq 0$ the simultaneous equations (2.6) have a unique non-trivial solution $\{C_k^n\}_{k=0}^{n}$, we have $\Delta_n(\phi) \neq 0$ ($n \geq 0$). Finally, we have that $K_n > 0$ for $n \geq 0$ if and only if $\Delta_n(\phi) > 0$ for $n \geq 0$. □

By various representation results like Boas' moment theorem [2] or Duran's generalization [3] of Boas' Theorem, any moment functional σ will have an integral representation of the form

$$\langle \sigma, \pi(x) \rangle = \int_{\mathbb{R}} \pi(x)\, d\sigma_0(x) \qquad (\pi(x) \in \mathcal{P}),$$

or

$$\langle \sigma, \pi(x) \rangle = \int_{\mathbb{R}} \pi(x) w_\sigma(x)\, dx \qquad (\pi(x) \in \mathcal{P}),$$

where $\sigma_0(x)$ is a function of bounded variation on $\mathbb{R}$ (so that $d\sigma_0$ is, in general, a signed Borel measure on $\mathbb{R}$) and $w_\sigma(x)$ is a C^∞-function of the Schwartz class. With these representations, the symmetric bilinear form given in (1.1) can be written as

$$(2.7) \qquad \phi(p,q) := (p,q)_N = \sum_{k=0}^{N} \langle \sigma_{(k)}, p^{(k)} q^{(k)} \rangle,$$

where $\sigma_{(k)}$ are moment functionals. As we shall see, it is more advantageous for us to use this abstract notation involving moment functionals instead of using one of the above integral representations of moment functionals.

In case the symmetric bilinear form $\phi(\cdot,\cdot)$ in (2.7) is quasi-definite (respectively, positive-definite), we call a corresponding PS $\{P_n(x)\}_{n=0}^{\infty}$ in Lemma 2.1 a Sobolev-Tchebycheff polynomial system (STPS) (respectively, a Sobolev orthogonal polynomial system (SOPS)) of order N relative to $\phi(\cdot,\cdot)$. When $N = 0$, we simply call $\{P_n(x)\}_{n=0}^{\infty}$ a Tchebycheff polynomial system(TPS) or an orthogonal polynomial system (OPS) relative to $\sigma_{(0)}$ and say that $\sigma_{(0)}$ is an orthogonalizing moment functional of $\{P_n(x)\}_{n=0}^{\infty}$.

Since a PS is a basis for $\mathcal{P}$, any PS $\{\phi_n(x)\}_{n=0}^{\infty}$ determines a momentfunctional σ (uniquely up to a non-zero constant multiple), called a canonical moment functional (see [11]) for $\{\phi_n(x)\}_{n=0}^{\infty}$, by the conditions

$$(2.8) \qquad \langle\sigma, \phi_0(x)\rangle \neq 0 \quad \text{and} \quad \langle\sigma, \phi_n(x)\rangle = 0 \qquad (n \geq 1).$$

Note that if a PS $\{P_n(x)\}_{n=0}^{\infty}$ is a TPS relative to σ, then σ must be a canonical moment functional for $\{P_n(x)\}_{n=0}^{\infty}$.

LEMMA 2.2. *Suppose that the symmetric bilinear form $\phi(\cdot,\cdot)$ in (2.7) is quasi-definite and let $\{P_n(x)\}_{n=0}^{\infty}$ be an STPS relative to$\phi(\cdot,\cdot)$. Then $\sigma_{(0)}$ must be a canonical moment functional for $\{P_n(x)\}_{n=0}^{\infty}$. In particular, $\langle\sigma_{(0)}, 1\rangle \neq 0$.*

Proof. This follows immediately from the orthogonality (2.3) by taking $m = 0$. □

Now we introduce some formal calculus on moment functionals. For a moment functional σ and a polynomial $\pi(x)$, we let σ', the derivative of σ, and $\pi\sigma$, multiplication of σ by a polynomial, be the moment functionals defined by

$$\langle\sigma', p(x)\rangle = -\langle\sigma, p'(x)\rangle \qquad (p(x) \in \mathcal{P})$$

and

$$\langle\pi(x)\sigma, p(x)\rangle = \langle\sigma, \pi(x)p(x)\rangle \qquad (p(x) \in \mathcal{P}).$$

It is then easy to obtain the following Leibnitz rule for any moment functional σ and polynomial $\pi(x)$:

$$(\pi(x)\sigma)' = \pi'(x)\sigma + \pi(x)\sigma'. \tag{2.9}$$

LEMMA 2.3. *Let σ be a moment functional and $\pi(x)$ a polynomial. Then*

(i) *$\sigma = 0$ if and only if $\sigma' = 0$.*

(ii) *If σ is quasi-definite and $\pi(x)\sigma = 0$, then $\pi(x) \equiv 0$.*

Proof. (i) If $\sigma = 0$, then $\langle\sigma', x^n\rangle = -n\langle\sigma, x^{n-1}\rangle = 0$ for any integer $n \geq 0$ so that $\sigma' = 0$. Conversely, if $\sigma' = 0$, then $\langle\sigma, x^n\rangle = \langle\sigma, \frac{1}{n+1}(x^{n+1})'\rangle = \frac{-1}{n+1}\langle\sigma', x^{n+1}\rangle = 0$ for any integer $n \geq 0$ so that $\sigma = 0$.

(ii) Assume that σ is quasi-definite and $\pi\sigma = 0$. Let $\{P_n(x)\}_{n=0}^{\infty}$ be a TPS relative to σ, satisfying the orthogonality condition $\langle\sigma, P_m P_n\rangle = K_n\delta_{mn}$ (m and $n \geq 0$) with $K_n \neq 0$. Write $\pi(x) = \sum_{k=0}^{N} C_k P_k(x)$, where $N = deg(\pi) \geq 0$ and $C_N \neq 0$. Then we have

$$0 = \langle\pi(x)\sigma, P_N\rangle = \sum_{k=0}^{N} C_k\langle\sigma, P_k P_N\rangle = C_N K_N,$$

so that $C_N = 0$, which is a contradiction. Hence $\pi(x) \equiv 0$. □

3. Polynomials satisfying second-order differential equations

Consider a second order differential equation of the form

$$L[y](x) = \ell_2(x)y''(x) + \ell_1(x)y'(x) = \lambda_n y(x), \tag{3.1}$$

where $\ell_2(x)$ and $\ell_1(x)$ are real-valued functions independent of the parameter n and λ_n is a real constant depending on n. S. Bochner [1] observed that if the differential equation (3.1) has a PS $\{P_n(x)\}_{n=0}^{\infty}$ of solutions, then its coefficients must be of the form

(3.2)

$$\ell_i(x) = \sum_{j=0}^{i} \ell_{ij}x^j \quad (i = 1, 2); \quad \lambda_n = n(n-1)\ell_{22} + n\ell_{11} \quad (n \geq 0),$$

where ℓ_{ij} are real constants with $\ell_{11}^2 + \ell_{22}^2 \neq 0$. From now on, we always assume that the coefficients of the differential equation (3.1) are the ones given in (3.2).

DEFINITION 3.1. (Krall and Sheffer [6]) The differential expression $L[\cdot]$ in (3.1) is called admissible if $\lambda_m \neq \lambda_n$ for $m \neq n$.

By direct calculation, it is easy to see that the differential equation (3.1) has a unique monic polynomial solution of degree n for each integer $n \geq 0$ except possibly for a finite number of values of n and for those exceptional values of n, there will be either no polynomial solution of degree n or infinitely many monic polynomial solutions of degree n to (3.1).

LEMMA 3.1. *For the differential equation* (3.1), *the following statements are equivalent.*

(i) $L[\cdot]$ *is admissible ;*
(ii) $\lambda_n = n(n-1)\ell_{22} + n\ell_{11} \neq 0$ *for* $n \geq 1$;
(iii) $\ell_{11} \notin \{-n\ell_{22} \mid n = 0, 1, \dots\}$;
(iv) *For each* $n \geq 0$, *the differential equation* (3.1) *has a unique monic polynomial solution of degree* n.

Proof. See Lemma 2.2 in [9]. □

DEFINITION 3.2. (Krall and Sheffer [6]) A PS $\{P_n(x)\}_{n=0}^{\infty}$ is called a weak Tchebycheff polynomial system (WTPS) if there is a non-trivial moment functional σ such that

$$\langle \sigma, P_m P_n \rangle = 0 \quad \text{for} \quad m \neq n. \tag{3.3}$$

In this case, we say that $\{P_n(x)\}_{n=0}^{\infty}$ is a WTPS relative to σ.

If $\{P_n(x)\}_{n=0}^{\infty}$ is a WTPS relative to σ, then σ is a canonical moment functional of $\{P_n(x)\}_{n=0}^{\infty}$ and $\langle \sigma, P_n^2(x) \rangle$ may or may not be zero for $n \geq 1$ (but $\langle \sigma, P_0^2(x) \rangle \neq 0$). We now discuss the orthogonality of polynomial solutions of the differential equation (3.1).

LEMMA 3.2. *If the differential equation* (3.1) *has a PS* $\{\phi_n(x)\}_{n=0}^{\infty}$ *of solutions, then any canonical moment functional* σ *of* $\{\phi_n(x)\}_{n=0}^{\infty}$ *must satisfy*

$$(\ell_2(x)\sigma)' - \ell_1(x)\sigma = 0, \tag{3.4}$$

which is equivalent to the recurrence relation
(3.5)

$$(n\ell_{22}+\ell_{11})\sigma_{n+1}+(n\ell_{21}+\ell_{10})\sigma_n+n\ell_{20}\sigma_{n-1}=0 \qquad (n\geq 0\ ;\ \sigma_{-1}=0),$$

where $\{\sigma_n\}_{n=0}^{\infty}$ *are the moments of* σ.

Proof. See Lemma 3.3 in [8]. □

We call the functional equation (3.4) the weight equation for the differential expression $L[\cdot]$ in (3.1), while (3.5) is called the moment equation for $L[\cdot]$ (in the L^2-sense).

REMARK 3.1. The weight equation (3.4) or equivalently, the moment equation (3.5) has at most two linearly independent solutions. It is now easy to see that the weight equation (3.4) has only one linearly independent solution (which may or maynot be quasi-definite) if the differential expression $L[\cdot]$ in (3.1) is admissible.

PROPOSITION 3.3. *If the differential equation* (3.1) *has a TPS* $\{P_n(x)\}_{n=0}^{\infty}$ *of solutions, then* $L[\cdot]$ *is admissible.*

Proof. See Proposition 3.5 in [8] □

Whether the differential operator $L[\cdot]$ is admissible or not, polynomial solutions of (3.1) have certain orthogonality. To be precise, we have the following.

PROPOSITION 3.4. *Let* $L[\cdot]$ *be the differential expression in* (3.1). *If* $L[p]=\lambda p$ *and* $L[q]=\mu q$ *for some* p, $q\in\mathcal{P}$ *and distinct real numbers* λ *and* μ, *then*

$$\langle\sigma, pq\rangle=0$$

for any solution σ *to the weight equation* (3.4). *In particular, if* $L[\cdot]$ *is admissible and* $\{P_n(x)\}_{n=0}^{\infty}$ *is a PS of solutions to* (3.1), *then* $\{P_n(x)\}_{n=0}^{\infty}$ *is a WTPS.*

Proof. The first part of this proposition is a special case of Proposition 4.4 below with $\sigma_{(k)}=0$, $k=1,2,\cdots,N$. The second part then follows immediately since the weight equation has only one linearly independent solution when $L[\cdot]$ is admissible (see Remark 3.1). □

THEOREM 3.5. *The differential equation* (3.1) *has a TPS of solutions if and only if the weight equation* (3.4) *has only one linearly independent solution* σ *which is quasi-definite.*

Proof. See Theorem 3.9 in [8]. □

THEOREM 3.6. *Assume that the differential equation* (3.1) *has a PS* $\{P_n(x)\}_{n=0}^{\infty}$ *of solutions. If* $\{P_n(x)\}_{n=0}^{\infty}$ *is not a TPS, then for any solution* σ *of the weight equation* (3.4), *there is an integer* $m \geq 0$ *such that*

$$\langle \sigma, P_n^2 \rangle = 0 \quad \text{for all} \quad n \geq m+1.$$

Proof. See Theorem 3.12 in [8]. □

4. Necessary and sufficient conditions and classifications

We first obtain necessary and sufficient conditions for an STPS relative to $\phi(\cdot,\cdot)$ in (2.7) to satisfy a second-order differential equation (3.1), of which the coefficients are given by (3.2).

LEMMA 4.1. *For* $N+1$ *moment functionals* $\sigma_{(k)}$, $k = 0, 1, \cdots N$, *and an integer* j, *let*

$$\begin{aligned} A_{k,j} =& \left[\binom{k+2}{j} - \binom{k}{j-2}\right]\ell_2\sigma_{(k)}^{(k-j+2)} + \left[2\binom{k+1}{j} - (k+j-1)\binom{k}{j-1}\right] \\ &\times \ell_2'\sigma_{(k)}^{(k-j+1)} + \left[\binom{k}{j} - \frac{(k+j)(k+j-1)}{2}\binom{k}{j}\right]\ell_2''\sigma_{(k)}^{(k-j)} \\ &- \left[\binom{k+1}{j} + \binom{k}{j-1}\right]\ell_1\sigma_{(k)}^{(k-j+1)} - \left[\binom{k}{j} + (k+j)\binom{k}{j}\right]\ell_1'\sigma_{(k)}^{(k-j+1)}, \end{aligned}$$

where $\binom{a}{b} = 0$ *if* $b < 0$ *or* $a < b$. *Then* $A_{k,j} = 0$ *for* $j < 0$ *and* $j \geq k+2$ *and*

$$A_{k,j} = \frac{k+j+1}{k-j+1}\binom{k}{j}A_{k,k+1}^{(k-j+1)} \quad \text{for all} \quad 0 \leq j \leq k. \tag{4.1}$$

Proof. It is easy to check by differentiating and rearranging the summation. □

THEOREM 4.2. *For a symmetric bilinear form $\phi(\cdot,\cdot)$ in (2.7), the following statements are equivalent.*

(i) *The differential expression $L[\cdot]$, given in (3.1), is symmetric on polynomials relative to $\phi(\cdot,\cdot)$, that is,*

$$\phi(L[p],q) = \phi(p, L[q]) \qquad (p,q \in \mathcal{P}). \tag{4.2}$$

(ii) *For $k = 0,1,\cdots N$, the moment functional $\sigma_{(k)}$ satisfies the functional equation*

$$\ell_2(x)\sigma'_{(k)} - [(k-1)\ell'_2(x) + \ell_1(x)]\sigma_{(k)} = 0. \tag{4.3}$$

(iii) *The moments of $\phi(\cdot,\cdot)$, $\sigma_{(k)}$, $k = 0,1,\cdots,N$, given by $\phi_{mn} := \phi(x^m, x^n)$, $\sigma^n_{(k)} := \langle\sigma_{(k)}, x^n\rangle$, ($m$ and $n \geq 0$), satisfy the equations*

$$\phi_{mn} = \sum_{k=0}^{N}(m,k)(n,k)\sigma^{m+n-2k}_{(k)}; \tag{4.4}$$

$$[(n+2k)\ell_{22} + \ell_{11}]\sigma^{n+1}_{(k)} + [(n+k)\ell_{21} + \ell_{10}]\sigma^n_{(k)} + n\ell_{20}\sigma^{n-1}_{(k)} = 0, \tag{4.5}$$

where $k = 0,1,\cdots,N$, $(a,b) = a(a-1)\cdots(a-b+1)$, and $\sigma^n_{(k)} = 0$ for $n < 0$.

Furthermore if $\phi(\cdot,\cdot)$ is quasi-definite and $\{P_n(x)\}_{n=0}^{\infty}$ is an STPS relative to $\phi(\cdot,\cdot)$, then statements (i), (ii), (iii) *are all equivalent to*

(iv) *$\{P_n(x)\}_{n=0}^{\infty}$ satisfies the differential equation* (3.1).

Proof. (i) $\Leftrightarrow$ (ii) : It is easy to check the following identities : for p, $q \in \mathcal{P}$

$$\phi(L[p],q) = \sum_{k=0}^{N}\langle\sigma_{(k)}, L(p)^{(k)}q^{(k)}\rangle = \sum_{k=0}^{N}(-1)^k\langle L^+[(q^{(k)}\sigma_{(k)})^{(k)}], p\rangle$$

and

$$\phi(p, L[q]) = \sum_{k=0}^{N}\langle\sigma_{(k)}, p^{(k)}L[q]^{(k)}\rangle = \sum_{k=0}^{N}(-1)^k\langle(L[q]^{(k)}\sigma_{(k)})^{(k)}, p\rangle,$$

where $L^+[\cdot]$ is the Lagrangian adjoint of $L[\cdot]$ defined by

$$L^+[y](x) = (\ell_2(x)y(x))'' - (\ell_1(x)y(x))'.$$

Hence, the equation (4.2) is equivalent to

$$\sum_{k=0}^{N}(-1)^k\{L^+[(q^{(k)}\sigma_{(k)})^{(k)}] - (L[q]^{(k)}\sigma_{(k)})^{(k)}\} = 0 \qquad (q \in \mathcal{P}), \tag{4.6}$$

which, when written out and simplified, yields, for all $q \in \mathcal{P}$,

$$\begin{aligned}
0 = &\sum_{k=0}^{N}(-1)^k\Bigg[\sum_{j=0}^{k}\binom{k}{j}\ell_2''\sigma_{(k)}^{(k-j)}q^{(k+j)} + \sum_{j=0}^{k+1}\binom{k+1}{j}2\ell_2'\sigma_{(k)}^{(k-j+1)}q^{(k+j)} \\
&+ \sum_{j=0}^{k+2}\binom{k+2}{j}\ell_2\sigma_{(k)}^{(k-j+2)}q^{(k+j)} - \sum_{j=0}^{k+1}\binom{k+1}{j}\ell_1\sigma_{(k)}^{(k-j+1)}q^{(k+j)} \\
&- \sum_{j=0}^{k}\binom{k}{j}\{\ell_1'\sigma_{(k)}^{(k-j)}q^{(k+j)} + \sigma_{(k)}^{(k-j)}(\ell_2 q'')^{(k+j)} + \sigma_{(k)}^{(k-j)}(\ell_1 q')^{(k+j)}\}\Bigg] \\
= &\sum_{k=0}^{N}(-1)^k\sum_{j=0}^{k+2}A_{k,j}q^{(k+j)} \\
= &\sum_{m=0}^{2N+2}\left(\sum_{k=0}^{N}(-1)^k A_{k,m-k}\right)q^{(m)} = \sum_{m=0}^{2N+1}\left(\sum_{k=0}^{N}(-1)^k A_{k,m-k}\right)q^{(m)}
\end{aligned}$$

where $A_{k,j}$ are the same as in Lemma 4.1. In the last two equalities above, we used the fact that $A_{k,j} = 0$ for $j < 0$ and $j \geq k+2$. Hence, the statement (i) is equivalent to

(4.7)

$$\begin{aligned}
0 = &\sum_{k=0}^{N}(-1)^k A_{k,m-k} = \sum_{k=[\frac{m}{2}]}^{min(m,N)}(-1)^k A_{k,m-k} \\
= &\begin{cases} \sum_{k=r}^{min(2r,N)}(-1)^k\frac{2r+1}{2k-2r+1}\binom{k}{2r-k}A_{k,k+1}^{(2k-2r+1)}, & \text{if } m = 2r \text{ is even} \\[2ex] (-1)^k A_{r,r+1} + \sum_{k=r+1}^{min(2r+1,k)}(-1)^k\frac{2r+2}{2k-2r}\binom{k}{2r+k}A_{k,k+1}^{(2k-2r)}, & \\ \qquad\qquad \text{if } m = 2r+1 \text{ is odd} \end{cases}
\end{aligned}$$

.

for $m = 0, 1, \cdots, 2N + 1$, where we use (4.1). By Lemma 2.3 (i), we can see inductively that the equations in (4.7) are equivalent to
(4.8)
$$A_{k,k+1} = 2\left[\ell_2(x)\sigma'_{(k)} - ((k-1)\ell'_2(x) + \ell_1(x))\sigma_{(k)}\right] = 0, \quad k = 0, 1, \cdots, N.$$

(ii) ⇔ (iii) : The equation (4.4) follows from the definition of $\phi(\cdot,\cdot)$. The equation (4.5) is just a restatement of the equation (4.3) in terms of the moments of $\sigma_{(k)}$. Finally assume that $\{P_n(x)\}_{n=0}^{\infty}$ is an STPS relative to a quasi-definite symmetric bilinear form $\phi(\cdot,\cdot)$.

(i) ⇒ (iv) : Since $L[P_n](x)$ is a polynomial of degree $\leq n$, we may write
$$L[P_n](x) = \sum_{j=0}^{n} C_{n,j} P_j(x)$$
for some real constants $C_{n,j}$, $j = 0, 1, \ldots, n$. Then for $k = 0, 1, \ldots, n-1$
$$C_{n,k}\phi(P_k, P_k) = \sum_{j=0}^{n} C_{n,j}\phi(P_j, P_k) = \phi(L[P_n], P_k) = \phi(P_n, L[P_k]) = 0$$
since $L[P_k](x)$ is a polynomial of degree $\leq k$. Hence, $C_{n,k} = 0$ for $k = 0, 1, \ldots, n-1$ and so $L[P_n](x) = C_{n,n}P_n(x) = \lambda_n P_n(x)$ by comparing the coefficients of x^n on both sides.

(iv) ⇒ (i) : If $L[P_n] = \lambda_n P_n$ for every integer $n \geq 0$, then we have for all integers m and $n \geq 0$
$$\phi(L[P_m], P_n) - \phi(P_m, L[P_n]) = (\lambda_m - \lambda_n)\phi(P_m, P_n) = 0.$$
We now have (4.2) by linearity since $\{P_n(x)\}_{n=0}^{\infty}$ is a basis for $\mathcal{P}$. □

We call the functional equations (4.3) the Sobolev weight equations for $L[\cdot]$ relative to $\phi(\cdot,\cdot)$ and the equations (4.5) are called the Sobolev moment equations for $L[\cdot]$ relative to $\phi(\cdot,\cdot)$.

As an immediate consequence of Theorem 4.2, we have the following generalization of Krall's theorem for classical orthogonal polynomials corresponding to the case $N = 0$.

COROLLARY 4.3. *Let $\phi(\cdot,\cdot)$ be a symmetric bilinear form given in (2.7). Then there is an STPS $\{P_n(x)\}_{n=0}^{\infty}$ relative to $\phi(\cdot,\cdot)$, which satisfies the differential equation (3.1) if and only if*

(i) *$\phi(\cdot,\cdot)$ is quasi-definite and*

(ii) *for each k, the moments $\{\sigma_{(k)}^n\}_{n=0}^{\infty}$ of $\sigma_{(k)}$ satisfies the Sobolev moment equation (4.3).*

Proof. This follows immediately from Lemma 2.1 and Theorem 4.2. □

REMARK 4.1. If we differentiate (3.1) with respect to x and set $y^{(k)}(x) = z(x)$, $k = 1, 2, \cdots N$, then we obtain second-order differential equations :

$$M_k[z](x) = \ell_2(x)z''(x) + (k\ell_2'(x) + \ell_1(x))z'(x) = \mu_{n+k}z(x), \tag{4.9}$$

where $\mu_{n+k} = \lambda_{n+k} - k\ell_1'(x) - \frac{k(k-1)}{2}\ell_2''(x)$. It is interesting and useful to note that the equations (4.3), are the weight equations for $M_k[\cdot]$, $k = 1, 2, \cdots N$, in the L^2-sense.

REMARK 4.2. When $\ell_2(x) \equiv 0$, the differential equation (3.1) reduces to the first-order equation

$$(\ell_{11}x + \ell_{10})y'(x) = n\ell_{11}y(x)$$

which can have a PS of solutions only when $\ell_{11} \neq 0$. In this case, the Sobolev weight equations (4.3) become

$$(\ell_{11}x + \ell_{10})\sigma_{(k)} = 0 \quad \text{for} \quad k = 0, 1, \cdots N,$$

of which the general solutions are

$$\sigma_{(k)} = c_k\delta(x + \ell_{10}/\ell_{11}),$$

where c_k's are constants. When these are substituted into (2.7), the corresponding symmetric bilinear form $\phi(\cdot,\cdot)$ cannot be quasi-definite. Consequently, by Corollary 4.3, the above first-order differential equation cannot have an STPS of solutions ; in particular, it cannot have a TPS of solutions.

Now we can prove the following which extends Proposition 3.4.

PROPOSITION 4.4. *Let $L[\cdot]$ be the differential expression in (3.1) and let $\phi(\cdot,\cdot)$ be the symmetric bilinear form in (2.7). If $L[p] = \lambda p$ and $L[q] = \mu q$ for some p, $q \in \mathcal{P}$ and distinct real numbers λ and μ, then*

$$\phi(p, q) = 0$$

for anysolutions $\sigma_{(k)}$ of the Sobolev weight equations (4.3) for $k = 0, 1, \cdots N$.

Proof. This follows from Theorem 4.2 and the identity

$$(\lambda - \mu)\phi(p, q) = \phi(\lambda p, q) - \phi(p, \mu q) = \phi(L[p], q) - \phi(p, L[q]). \quad \square$$

THEOREM 4.5. *Assume that the differential equation (3.1) has a $PS\{P_n(x)\}_{n=0}^{\infty}$ of solutions, which is an STPS relative to the bilinear form $\phi(\cdot,\cdot)$ in (2.7). If $\sigma_{(k)}$ is quasi-definite for some k, then*

(i) *$\{P_n^{(k)}(x)\}_{n=k}^{\infty}$ is a classical TPS relative to $\sigma_{(k)}$ satisfying the differential equation (4.9) ;*
(ii) *$M_i[\cdot]$ in (4.9) is admissible for all $i \geq k$;*
(iii) *For $i \geq k$, $\sigma_{(i)} = c_i(\ell_2(x))^{i-k}\sigma_{(k)}$ for some real constant c_i so that $\sigma_{(i)} = 0$ or $\sigma_{(i)}$ is quasi-definite ;*
(iv) *For $i < k$, $(\ell_2(x))^{k-i}\sigma_{(i)} = c_i\sigma_{(k)}$ for some real constant c_i so that $(\ell_2(x))^{k-i}\,\sigma_{(i)} = 0$ or $(\ell_2(x))^{k-i}\sigma_{(i)}$ is quasi-definite ;*
(v) *For each $i \geq k$, $\{P_n^{(i)}(x)\}_{n=i}^{\infty}$ is a TPS relative to $\sigma_{(i)}$ if $\sigma_{(i)} \neq 0$.*

Proof. (i) We may assume that each $P_n(x)$ is monic. Since $\{P_n(x)\}_{n=0}^{\infty}$ satisfies differential equation (3.1), it follows that $\{\frac{P_{n+k}^{(k)}(x)}{(n+k,k)}\}_{n=0}^{\infty}$ is a monic PS which satisfies the differential equation given by (4.9). Let $\{Q_n(x)\}_{n=0}^{\infty}$ be the monic TPS relative to $\sigma_{(k)}$. Since $\sigma_{(k)}$ satisfies the weight equation given in (4.3), by Theorem 4.2, $Q_n(x)$ satisfies the differential equation (4.9) and so is a classical TPS. By Proposition 3.3, the differential equation (4.9) is admissible. Hence, by Lemma 3.1, we must have $\frac{P_{n+k}^{(k)}(x)}{(n+k,k)} = Q_n(x)$, $n \geq 0$.

(ii) Since $M_k[\cdot]$ has a TPS $\{Q_n(x)\}_{n=0}^{\infty}$ of solutions, $M_k[\cdot]$ is admissible by Proposition 3.3. Now suppose that $M_i[\cdot]$ is not admissible for some $i > k$. Then by Lemma 3.1,

$$\lambda_{n+i} - i\ell_1'(x) - \frac{i(i-1)}{2}\ell_2''(x) = 0, \quad \text{for some} \quad n \geq i.$$

This implies that

$$\ell_{11} = -(n + 2i - 1)\ell_{22} \quad \text{for some} \quad n \geq i.$$

This contradicts that $M_k[\cdot]$ is admissible (i.e. $\ell_{11} \notin \{-n\ell_{22},\ n = k, k+1, \cdots\}$).

(iii) Since $M_i[\cdot]$ for $i \geq k$ in (4.9) is admissible, the moment equation associated to (4.9) is uniquely solvable by Remark 3.1. Define $\tilde{\sigma}_{(i)} = (\ell_2(x))^{i-k}\sigma_{(k)}$, then $\tilde{\sigma}_{(i)} \neq 0$ since $\ell_2(x) \not\equiv 0$ and $\sigma_{(k)}$ is quasi-definite (see Lemma 2.1). Moreover $\tilde{\sigma}_{(i)}$ satisfies the weight equation (4.3) for $k = i$. By the unique solvability, we have $\sigma_{(i)} = c_i(\ell_2(x))^{i-k}\sigma_{(k)}$ for some real constant c_i. Since $\{P_{n+k}^{(k)}(x)\}_{n=0}^{\infty}$ is a classical TPS relative to $\sigma_{(k)}$, $\{P_{n+i}^{(i)}(x)\}_{n=0}^{\infty}$, $i \geq k$, is also classical TPS satisfying the differential equation (4.9). Since $(\ell_2(x))^{(i-k)}\sigma_{(k)}$ is the unique solution of the weight equation (4.3) corresponding to $M_i[\cdot]$ so that it must be quasi-definite.

(iv) Since $M_k[\cdot]$ in (4.9) is admissible, the weight equation (4.3) is uniquely solvable. By the same reasoning as in (iii) we have $\sigma_{(k)} = c_i(\ell_2(x))^{k-i}\sigma_{(i)}$ for some real constant c_i.

(v) From (iii) we may assume that $\{Q_n(x)\}_{n=0}^{\infty}$ is a TPS relative to $\sigma_{(i)}$ if $\sigma_{(i)} \neq 0$. Since $\sigma_{(i)}$ satisfies the weight equation (4.3) for $k = i$, $\{Q_n(x)\}_{n=0}^{\infty}$ satisfies the differential equation (4.9) for $k = i$. By the admissibility, the differential equation (4.9) has a unique monic PS of solutions

$$Q_n(x) = \frac{P_{n+i}^{(i)}(x)}{(n+i, i)},$$

which is orthogonal relative to $\sigma_{(i)}$. □

As an application of Theorem 4.2 and Theorem 4.5, we now give the complete classification of STPS's relative to $(\cdot,\cdot)_2$ satisfying a second order differential equation of the form (3.1). Denoting $\sigma_{(0)}$, $\sigma_{(1)}$, $\sigma_{(2)}$

in (2.7) by σ, τ, ω respectively and setting $\sigma_{(k)} = 0$ for $k \geq 3$, we have a symmetric bilinear form

$$(4.10)\quad \phi(p,q) := (p,q)_2 = \langle \sigma, pq \rangle + \langle \tau, p'q' \rangle + \langle \omega, p''q'' \rangle \qquad (p,q \in \mathcal{P}).$$

In the following, we shall assume $\ell_2(x) \not\equiv 0$ (see Remark 4.2) and $\deg(\ell_1) \neq 0$ (i.e.,$\ell_1(x)$ is not a non-zero constant) since if $\deg(\ell_1) = 0$, then the differential equation (3.1) has no polynomial solution of degree one.

Concerning the symmetric bilinear form $\phi(\cdot,\cdot)$ in (4.10), there arise the following four cases :

(i) type A : σ is quasi-definite ;
(ii) type B : τ is quasi-definite but σ is not quasi-definite ;
(iii) type C : σ and τ are not quasi-definite but ω is quasi-definite ;
(iv) type D : σ, τ, ω are not quasi-definite.

We note that the symmetric bilinear form $\phi(\cdot,\cdot)$ in (2.7) can bequasi-definite even though $\sigma_{(k)}$ are not quasi- definite for all $k = 0, 1, \cdots, N$. Indeed, Duran [4] produced the following example.

EXAMPLE 4.1. Define a moment functional σ by its moments $\{\sigma_n\}_{n=0}^{\infty}$:

$$\sigma_0 = 3, \quad \sigma_1 = 1, \quad \text{and} \quad \sigma_n = \frac{1}{n+1} = \int_0^1 x^n \, dx \qquad (n \geq 2).$$

Since $\Delta_2(\sigma) = 0$, σ is not quasi-definite. The moment functional $\tau = \frac{1}{8}\delta(x)$ is clearly not quasi-definite. However, the bilinear form $\phi(\cdot,\cdot)$ defined by

$$\phi(p,q) = \langle \sigma, pq \rangle + \frac{1}{8}\langle \tau, p'q' \rangle$$

is positive-definite. Indeed, for any $0 \not\equiv p \in \mathcal{P}$, we have

$$\phi(p,p) = \int_0^1 p^2(x)\,dx + (\sqrt{2}p(0) + \frac{1}{2\sqrt{2}}p'(0))^2 > 0.$$

Type A : σ is quasi-definite. In this case, any STPS relative to $\phi(\cdot,\cdot)$ in (4.10) must be a classical TPS relative to σ by Theorem 4.5

and there exist, up to a reallinear change of variable, six distinct types : Jacobi, Bessel, Laguerre, Hermite, twisted Jacobi and twisted Hermite polynomials (see [8]), which we list below for later use. In the following we use the notations

$$\binom{\alpha}{0} = 1 \quad \text{and} \quad \binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!},$$

for any complex number α and any integer $k \geq 1$.

Jacobi polynomials : Assume $\ell_2(x) = 0$ has two distinct real roots. Then, by a real linear change of variable, we may transform the differential equation (3.1) into

(4.11)
$$\begin{aligned} L[y](x) &= (1-x^2)y''(x) + [(\beta-\alpha) - (\alpha+\beta+2)x]y'(x) \\ &= -n(n+\alpha+\beta+1)y(x). \end{aligned}$$

From Lemma 3.1, we can see that the equation (4.11) is admissible if and only if $\alpha+\beta+1 \notin \{-1,-2,\dots\}$. When the equation (4.11) is admissible, it has a unique monic PS $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ of solutions, called the Jacobi polynomials, where

(4.12)
$$P_n^{(\alpha,\beta)}(x) = \binom{2n+\alpha+\beta}{n}^{-1} \sum_{k=0}^{n} \binom{n+\alpha}{k}\binom{n+\beta}{n-k}(x-1)^{n-k}(x+1)^k,$$

$n \geq 0$. The Jacobi PS $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ is a TPS (respectively, an OPS) if and only if $\alpha+\beta+1$, α, and $\beta \notin \{-1,-2,\dots\}$ (respectively, α and $\beta > -1$).

Bessel polynomials : Assume $\ell_2(x) = 0$ has a double real root. Then, by a real linear change of variable, we may transform the differential equation (3.1) into

(4.13) $$L[y](x) = x^2y''(x) + (\alpha x+\beta)y'(x) = n(n+\alpha-1)y(x).$$

From Lemma 3.1, we can see that the equation (4.13) is admissible if and only if $\alpha \notin \{0, -1, -2, \dots\}$. When the equation (4.13) is admissible, it has a unique monic PS $\{B_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ of solutions given by

(4.14)

$$B_n^{(\alpha,\beta)}(x) = \begin{cases} x^n & \text{if } \beta = 0 \\ \dfrac{\beta^n}{\Gamma(\alpha + 2n - 1)} \displaystyle\sum_{k=0}^{n} \frac{n!\Gamma(\alpha + n + k - 1)}{(n-k)!\,k!} \left(\frac{x}{\beta}\right)^k & \text{if } \beta \neq 0 \end{cases}$$

$n \geq 0$. When $\beta \neq 0$, we call $\{B_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ the Bessel PS. The PS $\{B_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ is a TPS (but never an OPS) if and only if $\alpha \notin \{0, -1, -2, \dots\}$ and $\beta \neq 0$.

Laguerre polynomials : Assume $\deg(\ell_2) = 1$. Then, by a real linear change of variable, we may transform the differential equation (3.1) into

$$L[y](x) = xy''(x) + (\alpha + 1 - x)y'(x) = -ny(x). \tag{4.15}$$

The equation (4.15) is admissible and so has a unique monic PS $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ of solutions, called the Laguerre polynomials, where

$$L_n^{(\alpha)}(x) = (-1)^n n! \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!}, \qquad n \geq 0. \tag{4.16}$$

The Laguerre PS $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ is a TPS (respectively, an OPS) if and only if $\alpha \notin \{-1, -2, \dots\}$ (respectively, $\alpha > -1$).

Hermite polynomials : Assume $\deg(\ell_2) = 0$ and $\ell_{11} < 0$. Then, by a real linear change of variable, we may transform the differential equation (3.1) into

$$L[y](x) = y''(x) - 2xy'(x) = -2ny(x). \tag{4.17}$$

The equation (4.17) is admissible and so has a unique monic PS $\{H_n(x)\}_{n=0}^{\infty}$ of solutions, called the Hermite polynomials, where

$$H_n(x) = n! \sum_{k=0}^{[n/2]} \frac{(-1)^k}{k!\,(n-2k)!} \frac{x^{n-2k}}{4^k}, \qquad n \geq 0. \tag{4.18}$$

The Hermite PS $\{H_n(x)\}_{n=0}^{\infty}$ is an OPS.

Twisted Hermite polynomials : Assume $\deg(\ell_2) = 0$ and $\ell_{11} > 0$. Then, by a real linear change of variable, we may transform the differential equation (3.1) into

$$L[y](x) = y''(x) + 2xy'(x) = 2ny(x). \tag{4.19}$$

The equation (4.19) is admissible and so has a unique monic PS $\{\check{H}_n(x)\}_{n=0}^{\infty}$ of solutions, called the twisted Hermite polynomials, where

$$\check{H}_n(x) = n! \sum_{k=0}^{[n/2]} \frac{1}{k!\,(n-2k)!} \frac{x^{n-2k}}{4^k}, \qquad n \geq 0. \tag{4.20}$$

The twisted Hermite PS $\{\check{H}_n(x)\}_{n=0}^{\infty}$ is a TPS (but never an OPS). In terms of Hermite polynomials, we have $\check{H}_n(x) = i^n H_n(-ix)$ $(n \geq 0)$; here $i = \sqrt{-1}$.

Twisted Jacobi polynomials : Assume $\ell_2(x) = 0$ has two complex conjugate roots. Then, by a real linear change of variable, we may transform the differential equation (3.1) into

$$L[y](x) = (1 + x^2)y''(x) + (bx + c)y'(x) = n(n + b - 1)y(x). \tag{4.21}$$

From Lemma 3.1, we can see that the equation (4.21) is admissible if and only if $b \notin \{0, -1, -2, \dots\}$. When the equation (4.21) is admissible, it has a unique monic PS $\{\check{P}_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ of solutions, called the twisted Jacobi polynomials, where

(4.22)

$$\check{P}_n^{(\alpha,\beta)}(x) = \binom{2n+\alpha+\beta}{n}^{-1} \sum_{k=0}^{n} \binom{n+\alpha}{k} \binom{n+\beta}{n-k} (x-i)^{n-k}(x+i)^k,$$

$n \geq 0$, where $ic = \beta - \alpha$ and $b = \alpha + \beta + 2$. The twisted Jacobi PS $\{\check{P}_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ is a TPS (but never an OPS) if and only if $b = \alpha + \beta + 2 \notin \{0, -1, -2, \dots\}$. In terms of Jacobi polynomials, we have $\check{P}_n^{(\alpha,\beta)}(x) = i^n P_n^{(\alpha,\beta)}(-ix)$ $(n \geq 0)$.

Type B : τ is quasi-definite but σ is not quasi- definite. From Theorem 4.5 we have

PROPOSITION 4.6. *Assume that the differential equation* (3.1) *has a PS* $\{P_n(x)\}_{n=0}^{\infty}$ *of solutions, which is an STPS relative to the bilinear form* $\phi(\cdot,\cdot)$ *in* (4.10). *If* τ *is quasi-definite, then*

(i) $\{P_n'(x)\}_{n=1}^{\infty}$ *is a classical TPS relative to* τ *satisfying the differential equation* (4.9) *for* $k = 1$;

(ii) $M_i[\cdot]$ *in* (4.9) *is admissible for* $i = 1, 2$;

(iii) $\omega = c_2\ell_2(x)\tau$ *for some real constants* c_2 *so that* $\omega = 0$ *or* ω *is quasi-definite;*

(iv) $\ell_2(x)\sigma = c_1\tau$ *for some real constant* c_1 *so that either* $\ell_2(x)\sigma = 0$ *or* $\ell_2(x)\sigma$ *is quasi-definite* ;

(v) $\{P_n''(x)\}_{n=2}^{\infty}$ *is a TPS relative to* $\ell_2(x)\tau$;

(vi) $\{P_n(x)\}_{n=0}^{\infty}$ *is a WTPS relative to* σ.

Proof. It is immediately obtained from Theorem 4.5 and the Sobolev orthogonality. □

Assume that $\{P_n(x)\}_{n=0}^{\infty}$ is an STPS as in Proposition 4.6 and that σ is not quasi-definite but τ is quasi-definite. Then, by Proposition 4.6, $\{P_n'(x)\}_{n=1}^{\infty}$ is a classical TPS relative to τ and $\{P_n(x)\}_{n=0}^{\infty}$ is a WTPS relative to σ. Hence, $\{P_n(x)\}_{n=0}^{\infty}$ is also orthogonal relative to the Sobolev inner product

$$(p,q)_1 := \langle\sigma, pq\rangle + c\langle\tau, p'q'\rangle,$$

where c is any non-zero constant satisfying $c \notin \{-\frac{\langle\sigma,P_n^2\rangle}{\langle\tau,(P_n')^2\rangle} \mid n \geq 1\}$. This is the case which is already considered in [8] and there are, up to a real linear change of variable, five such STPS's : three Jacobi types $\{P_n^{(-1,-1)}(x)\}_{n=0}^{\infty}$, $\{P_n^{(-1,\beta)}(x)\}_{n=0}^{\infty}(\beta \neq -1,-2,\cdots)$, $\{P_n^{(\alpha,-1)}(x)\}_{n=0}^{\infty}$ $(\alpha \neq -1,-2,\cdots)$; one Laguerre type $\{L_n^{(-1)}(x)\}_{n=0}^{\infty}$; one twisted Jacobi type $\{\tilde{P}_n^{(-1,-1)}(x)\}_{n=0}^{\infty}$.

Type C : ω is quasi-definite but σ, τ are not quasi-definite.

PROPOSITION 4.7. *Assume that the differential equation* (3.1) *has a PS* $\{P_n(x)\}_{n=0}^{\infty}$ *of solutions, which is an STPS relative to the bilinear form* $\phi(\cdot,\cdot)$ *in* (4.10). *If* ω *is quasi-definite, then*

(i) $\{P_n''(x)\}_{n=2}^{\infty}$ *is a classical TPS relative to* ω *satisfying the differential equation* (4.9) *for* $k = 2$;

(ii) $\ell_2^2(x)\sigma = c_1\omega$ *for some real constants* c_1 *so that either* $\ell_2^2(x)\sigma = 0$ *or* $\ell_2^2(x)\sigma$ *is quasi-definite ;*

(iii) $\ell_2(x)\tau = c_2\omega$ *for some real constants* c_2 *so that either* $\ell_2(x)\tau = 0$ *or* $\ell_2(x)\tau$ *is quasi-definite.*

Proof. It is immediately obtained from Theorem 4.5 and the Sobolev orthogonality. □

If $\ell_2(x)$ is a non-zero constant, then σ is also quasi-definite by Proposition 4.7. This case reduces to type A. Hence, in type C, we may assume $\deg(\ell_2) \geq 1$ so that there are four cases

$$\ell_2(x) = 1 - x^2,\ x^2,\ x,\ 1 + x^2.$$

In each case, we look for the conditions such that the Sobolev weight equations (4.3) have no quasi-definite moment functional solutions for $k \leq 1$ and have a quasi-definite moment functional solution for $k = 2$. To do this, we use the previous classifications of classical TPS's and Theorem 3.5.

Case C.1. Jacobi type ($\ell_2(x) = 1 - x^2$). The differential equation is (4.23)

$$(1 - x^2)y''(x) + ((\beta - \alpha) - (\alpha + \beta + 2)x)y'(x) = -n(n + \alpha + \beta + 1)y(x).$$

The Sobolev weight equations corresponding to (4.23) are

$$[(1 - x^2)\sigma]' - [(\beta - \alpha) - (\alpha + \beta + 2)x]\sigma = 0; \tag{4.24}$$

$$(1 - x^2)\tau' - [(\beta - \alpha) - (\alpha + \beta + 2)x]\tau = 0; \tag{4.25}$$

$$(1 - x^2)\omega' - [(\beta - \alpha) - (\alpha + \beta + 4)x]\omega = 0. \tag{4.26}$$

The equation (4.24) has no quasi-definite moment functional solution σ if and only if $\alpha + \beta + 1$ or α or $\beta \in \{-1, -2, \dots\}$, and the equation (4.25) has no quasi-definite moment functional solution τ if and only if $\alpha + \beta + 3$ or $\alpha + 1$, or $\beta + 1 \in \{-1, -2, \dots\}$, whereas the equation (4.26) has a quasi-definite moment functional solution ω if and only if $\alpha + \beta + 5$ and $\alpha + 2$ and $\beta + 2 \notin \{-1, -2, \cdots\}$. Hence, there arise six cases :

(i) $\alpha + \beta + 1 = -1$ and $\alpha = -2$ or $\beta = -2$;

(ii) $\alpha+\beta+1=-2$ and $\alpha=-2$ or $\beta=-2$;
(iii) $\alpha+\beta+3=-1$, or -2 with $\alpha\notin\{-3,-4,\cdots\}$ and $\beta\notin\{-3,-4,\cdots\}$;
(iv) $\alpha=-2$ and $\beta=-2$;
(v) $\alpha=-2$ and $\beta\notin\{-3,-4,\cdots\}$;
(vi) $\beta=-2$ and $\alpha\notin\{-2,-3,\dots\}$.

Case C.1.1. $\alpha+\beta+1=-1$ **and** $\alpha=-2$ **or** $\beta=-2$ In this case, we have $\alpha=-2$, $\beta=0$ or $\alpha=0$, $\beta=-2$. In both cases, $\ell_1(x)=\beta-\alpha$ is a non-zero constant so that the differential equation (4.23) has no polynomial solution of degree one.

Case C.1.2. $\alpha+\beta+1=-2$ **and** $\alpha=-2$ **or** $\beta=-2$. In this case, we have $\alpha=-2$, $\beta=-1$ or $\alpha=-1$, $\beta=-2$.
For $\alpha=-2$, $\beta=-1$, the differential equation (4.23) becomes

$$L[y](x)=(1-x^2)y''(x)+(1+x)y'(x)=-n(n-2)y(x). \tag{4.27}$$

The equation (4.27) is not admissible but has monicpolynomial solutions

$$P_n^{(-2,-1)}(x)=\binom{2n-3}{n}^{-1}\sum_{k=0}^{n}\binom{n-2}{k}\binom{n-1}{n-k}(x-1)^{n-k}(x+1)^k,\quad \text{for } n\neq 2$$

and $P_2^{(-2,-1)}(x)=x^2-2x+\gamma$, where γ is an arbitrary constant. The Sobolev weight equations corresponding to (4.27) are

$$[(1-x^2)\sigma]'-(1+x)\sigma=0; \tag{4.28}$$

$$(1-x^2)\tau'-(1+x)\tau=0; \tag{4.29}$$

$$(1-x^2)\omega'-(1-x)\omega=0. \tag{4.30}$$

The general solutions of (4.28), (4.29), and (4.30) are $\sigma=c_1\delta(x+1)$, $\tau=c_2\delta(x-1)$, and $\omega=c_3(1+x)H(1-x^2)$, where $c_i\,(i=1,2,3)$ are arbitrary constants and $H(x)$ is the Heaviside function. Hence, the corresponding bilinear form (4.10) is

$$\phi(p,q)=c_1p(-1)q(-1)+c_2p'(1)q'(1)+c_3\int_{-1}^{1}(1-x)p''(x)q''(x)\,dx.$$

Since ω is quasi-definite if and only if $c_3 \neq 0$, we may consider only the following two parameter family of bilinear forms (by setting $A = c_1/c_3$ and $B = c_2/c_3$) :

$$(4.31)\quad \phi(p,q) = Ap(-1)q(-1)+Bp'(1)q'(1)+\int_{-1}^{1}(1+x)p''(x)q''(x)\,dx.$$

PROPOSITION 4.8. *The bilinear form* $\phi(\cdot,\cdot)$ *in* (4.31) *is*

(i) *quasi-definite if and only if* $A \neq 0$ *and* $B \neq 0$;

(ii) *positive-definite if and only if* $A > 0$ *and* $B > 0$.

In either case, the monic STPS or SOPS relative to $\phi(\cdot,\cdot)$ *is* $\{P_n^{(-2,-1)}(x)\}_{n=0}^{\infty}$ *where* $P_2^{(-2,-1)}(x) = x^2 - 2x - 3$ *and squared norms are given by*

(4.32)

$$\phi(P_n^{(-2,-1)}(x), P_n^{(-2,-1)}(x)) = \begin{cases} A & \text{if } n = 0 \\ B & \text{if } n = 1 \\ [n(n-1)]^2 K_{n-2}(0,1) & \text{if } n \geq 2 \end{cases}$$

where

$$K_n(0,1) = \int_{-1}^{1}(1+x)[P_n^{(0,1)}(x)]^2\,dx = \frac{2^{2n+2}[(n+1)!(n)!]^2}{(2n+2)!(2n+1)!} \quad (n \geq 0).$$

Proof. Direct calculation shows that $\phi(P_2^{(-2,-1)}(x), P_0^{(-2,-1)}(x)) = 0$ only when $\gamma = -3$. Then Proposition 4.4 shows that $\phi(P_m^{(-2,-1)}(x), P_n^{(-2,-1)}(x)) = 0$ for $m \neq n$. Since we have

(i) $P_n^{(-2,-1)}(-1) = 0$ for all $n \geq 1$;

(ii) $[P_n^{(-2,-1)}]'(1) = 0$ for all $n \geq 2$;

(iii) $[P_n^{(-2,-1)}(x)]'' = n(n-1)P_{n-2}^{(0,1)}(x)$ for all $n \geq 2$;

(iv) $\langle(1+x)H(1-x^2), P_m^{(0,1)}P_n^{(0,1)}\rangle = K_n(0,1)\delta_{mn}$, m and $n \geq 0$,

we obtain (4.32) from which the result follows. □

For $\alpha = -1$, $\beta = -2$, the differential equation (4.23) becomes

$$(4.33)\quad L[y](x) = (1-x^2)y''(x) + (-1+x)y'(x) = -n(n-2)y(x).$$

The equation (4.33) is not admissible but has monic polynomials of solutions

$$P_n^{(-1,-2)}(x) = \binom{2n-3}{n}^{-1} \sum_{k=0}^{n} \binom{n-1}{k}\binom{n-2}{n-k}(x-1)^{n-k}(x+1)^k,$$
$$\text{for } \; n \neq 2$$

and $P_2^{(-1,-2)}(x) = x^2 + 2x + \gamma$, where γ is an arbitrary constant. The Sobolev weight equations corresponding to (4.33) are

$$[(1-x^2)\sigma]' - (-1+x)\sigma = 0; \tag{4.34}$$

$$(1-x^2)\tau' - (-1+x)\tau = 0; \tag{4.35}$$

$$(1-x^2)\omega' - (-1-x)\omega = 0. \tag{4.36}$$

The general solutions of (4.34), (4.35), and (4.36) are $\sigma = c_1\delta(x-1)$, $\tau = c_2\delta(x+1)$, and $\omega = c_3(1-x)H(1-x^2)$, where c_i $(i = 1,2,3)$ are arbitrary constants. As before we may consider only the following two parameter family of bilinear forms :

$$\phi(p,q) = Ap(1)q(1) + Bp'(-1)q'(-1) + \int_{-1}^{1}(1-x)p''(x)q''(x)\,dx. \tag{4.37}$$

Proposition 4.9. *The bilinear form $\phi(\cdot,\cdot)$ in (4.37) is*

(i) *quasi-definite if and only if $A \neq 0$ and $B \neq 0$;*

(ii) *positive-definite if and only if $A > 0$ and $B > 0$.*

In either case, the monic STPS or SOPS relative to $\phi(\cdot,\cdot)$ is $\{P_n^{(-1,-2)}(x)\}_{n=0}^{\infty}$ where $P_2^{(-1,-2)}(x) = x^2 + 2x - 3$ and squared norms are given by

(4.38)

$$\phi(P_n^{(-1,-2)}(x), P_n^{(-1,-2)}(x)) = \begin{cases} A & \text{if } n = 0 \\ B & \text{if } n = 1 \\ [n(n-1)]^2 K_{n-2}(1,0) & \text{if } n \geq 2 \end{cases}$$

where

$$K_n(1,0) = \int_{-1}^{1}(1-x)[P_n^{(1,0)}(x)]^2\,dx = \frac{2^{2n+2}[(n+1)!(n)!]^2}{(2n+2)!(2n+1)!} \quad (n \geq 0).$$

Proof. Direct calculation shows that $\phi(P_2^{(-1,-2)}(x), P_0^{(-1,-2)}(x)) = 0$ only when $\gamma = -3$. Then Proposition 4.4 shows that $\phi(P_m^{(-1,-2)}(x), P_n^{(-1,-2)}(x)) = 0$ for $m \neq n$. Since we have

(i) $P_n^{(-1,-2)}(1) = 0$ for all $n \geq 1$;

(ii) $[P_n^{(-1,-2)}]'(-1) = 0$ for all $n \geq 2$;

(iii) $[P_n^{(-1,-2)}(x)]'' = n(n-1)P_{n-2}^{(1,0)}(x)$ for all $n \geq 2$;

(iv) $\langle (1-x)H(1-x^2), P_m^{(1,0)}P_n^{(1,0)} \rangle = K_n(1,0)\delta_{mn}$, m and $n \geq 0$,

we obtain (4.38) from which the result follows. □

Case C.1.3 : $\alpha + \beta + 3 = -1$ or -2 with $\alpha \notin \{-3, -4, \cdots\}$ and $\beta \notin \{-3, -4, \cdots\}$.
For $\alpha + \beta + 3 = -1$, the differential equation (4.23) has a polynomial solution of degree 2 only when $\alpha = -2$. Hence, this case reduces to case C.1.4 below.
For $\alpha + \beta + 3 = -2$, the differential equation (4.23) has no polynomial solution of degree four.

Case C.1.4 : $\alpha = -2$, $\beta = -2$. The differential equation (4.23) becomes

$$L[y](x) = (1-x^2)y''(x) + 2xy'(x) = -n(n-3)y(x). \tag{4.39}$$

The equation (4.39) is not admissible but has monic polynomial solutions

$$P_n^{(-2,-2)}(x) = \binom{2n-4}{n}^{-1} \sum_{k=0}^{n} \binom{n-2}{k}\binom{n-2}{n-k}(x-1)^{n-k}(x+1)^k, \quad \text{for } n \neq 2, 3$$

and $P_2^{(-2,-2)}(x) = x^2 + \gamma_1 x + 1$, $P_3^{(-2,-2)}(x) = x^3 - 3x + \gamma_2$, where γ_1, γ_2 are arbitrary constants. The Sobolev weight equations corresponding to (4.39) are

$$[(1-x^2)\sigma]' - 2x\sigma = 0; \tag{4.40}$$

$$(1-x^2)\tau' - 2x\tau = 0; \tag{4.41}$$

$$(1-x^2)\omega' = 0. \tag{4.42}$$

We have two linearly independent solutions for σ, τ from (4.40) and (4.41) respectively and one linearly independent solution ω from (4.42). They are

$$\sigma^{(1)} = \delta(x-1) + \delta'(x-1), \quad \sigma^{(2)} = \delta(x+1) - \delta'(x+1);$$
$$\tau^{(1)} = \delta(x-1), \qquad \tau^{(2)} = \delta(x+1);$$
$$\omega = H(1-x^2).$$

As before we may consider only the following four parameter family of bilinear forms :
(4.43)

$$\begin{aligned}\phi(p,q) = A[p(1)q(1) - (p'(1)q(1) + p(1)q'(1))] + B[(p(-1)q(-1)\\ + (p'(-1)q(-1) + p(-1)q'(-1))] + Cp'(1)q'(1)\\ + Dp'(-1)q'(-1) + \int_{-1}^{1} p''(x)q''(x)\,dx\end{aligned}$$

PROPOSITION 4.10. *The bilinear form $\phi(\cdot,\cdot)$in (4.43) is*

(i) *quasi-definite if and only if $A+B \neq 0$, $A+B \neq C+D$ and $2(B-D)(C-A) - (A+B-C-D) \neq 0$;*
(ii) *positive- definite if and only if $A+B > 0$, $A+B < C+D$ and $2(B-D)(C-A) - (A+B-C-D) > 0$*

In either case, the monic STPS or SOPS relative to $\phi(\cdot,\cdot)$ is $\{P_n^{(-2,-2)}(x)\}_{n=0}^{\infty}$ where $P_2^{(-2,-2)}(x) = x^2+\gamma_1 x+1$ and $P_3^{(-2,-2)}(x) = x^3-3x+\gamma_2$, with $\gamma_1 = \frac{-2(A-B-C+D)}{A+B-C-D}$, $\gamma_2 = \frac{2(A-B)}{A+B}$ and squared norms are given by
(4.44)

$$\phi(P_n^{(-2,-2)}(x), P_n^{(-2,-2)}(x)) = \begin{cases} A+B & \text{if } n=0 \\ -A-B+C+D & \text{if } n=1 \\ E & \text{if } n=2 \\ [n(n-1)]^2 K_{n-2}(0,0) & \text{if } n \geq 3 \end{cases}$$

where $E = (C-A)(\gamma_1+2)^2 + (D-B)(\gamma_1-2)^2 + 8$ and

$$K_n(0,0) = \int_{-1}^{1} [P_n^{(0,0)}(x)]^2\,dx = \frac{2^{2n+1}(n!)^4}{(2n)!(2n+1)!} \quad (n \geq 0).$$

Conversely, with any choice of γ_1, γ_2, $\{P_n^{(-2,-2)}(x)\}_{n=0}^{\infty}$ is an STPS relative to $\phi(\cdot,\cdot)$ if A, B, C, and D are such that

$$A + B \neq 0, \quad A + B \neq C + D,$$
$$(C - A)(\gamma_1 + 2)^2 + (D - B)(\gamma_1 - 2)^2 + 8 \neq 0,$$
$$(A + B)\gamma_2 - 2(A - B) = 0,$$

and

$$(A + B - C - D)\gamma_1 + 2(A - B - C + D) = 0.$$

Proof. Direct calculation shows that $\phi(P_3^{(-2,-2)}(x), P_0^{(-2,-2)}(x)) = 0$ and $\phi(P_2^{(-2,-2)}(x), P_1^{(-2,-2)}(x)) = 0$. Then Proposition 4.4 shows that
$\phi(P_m^{(-2,-2)}(x), P_n^{(-2,-2)}(x)) = 0$ for $m \neq n$. Since we have

(i) $P_n^{(-2,-2)}(\pm 1) = 0$ for all $n \geq 4$;
(ii) $[P_n^{(-2,-2)}]'(\pm 1) = 0$ for all $n \geq 3$;
(iii) $[P_n^{(-2,-2)}(x)]'' = n(n-1)P_{n-2}^{(0,0)}(x)$ for all $n \geq 3$;
(iv) $\langle H(1 - x^2), P_m^{(0,0)} P_n^{(0,0)} \rangle = K_n(0,0)\delta_{mn}$, m and $n \geq 0$,

we obtain (4.44) from which the result follows. □

Case C.1.5 : $\alpha = -2$ and $\beta \notin \{-3, -4, \cdots\}$. The differential equation (4.23) becomes

(4.45) $L[y](x) = (1-x^2)y''(x) + ((\beta+2) - \beta x)y'(x) = -n(n+\beta-1)y(x).$

The equation (4.45) is admissible and so has a unique monic PS $\{P_n^{(-2,\beta)}(x)\}_{n=0}^{\infty}$ of solutions

$$P_n^{(-2,\beta)}(x) = \binom{2n+\beta-2}{n}^{-1} \sum_{k=0}^{n} \binom{n-2}{k} \binom{n+\beta}{n-k} (x-1)^{n-k}(x+1)^k,$$
$$(n \geq 0).$$

The Sobolev weight equations corresponding to (4.45) are

(4.46) $$[(1 - x^2)\sigma]' - [(\beta + 2) - \beta x]\sigma = 0;$$

(4.47) $$(1 - x^2)\tau' - [(\beta + 2) - \beta x]\tau = 0;$$

(4.48) $$(1 - x^2)\omega' - (\beta + 2)(1 - x)\omega = 0.$$

The general solutions of (4.46), (4.47), and (4.48) are $\sigma = c_1[\beta\delta(x-1) - 2\delta'(x-1)]$, $\tau = c_2\delta(x-1)$, $\omega = c_3(1+x)_+^{\beta+2}H(1-x)$, where c_i $(i = 1, 2, 3)$ are arbitrary constants. For information on the distribution x_+^α, we refer to Hörmander [5] and Morton and Krall [12]. As before we may consider only the following two parameter family of bilinear forms :

$$\begin{aligned}\phi(p,q) =& A[\beta p(1)q(1) + 2(p'(1)q(1) + p(1)q'(1))] \\ &+ Bp'(1)q'(1) + \langle (1+x)_+^{\beta+2}H(1-x), p''(x)q''(x)\rangle.\end{aligned} \tag{4.49}$$

PROPOSITION 4.11. *The bilinear form* $\phi(\cdot,\cdot)$ *in* (4.49) *is*

(i) *quasi-definite if and only if* $A\beta \neq 0$, $B - \frac{4}{\beta}A \neq 0$;

(ii) *positive-definite if and only if* $A\beta > 0$, $B - \frac{4}{\beta}A > 0$.

In either case, the monic STPS or SOPS relative to $\phi(\cdot,\cdot)$ *is* $\{P_n^{(-2,\beta)}(x)\}_{n=0}^\infty$ *and squared norms are given by*

$$\phi(P_n^{(-2,\beta)}(x), P_n^{(-2,\beta)}(x)) = \begin{cases} \beta A & \text{if } n = 0 \\ B - \frac{4}{\beta}A & \text{if } n = 1 \\ [n(n-1)]^2K_{n-2}(0,\beta+2) & \text{if } n \geq 2 \end{cases} \tag{4.50}$$

where

$$K_n(0,\beta+2) = \frac{2^{2n+\beta+2}[n!\Gamma(n+\beta+2)]^2}{\Gamma(2n+\beta+4)\Gamma(2n+\beta+3)} \quad (n \geq 0).$$

Proof. Proposition 4.4 shows that $\phi(P_m^{(-2,\beta)}(x), P_n^{(-2,\ beta)}(x)) = 0$ for $m \neq n$. Since we have

(i) $P_n^{(-2,\beta)}(1) = 0$ for all $n \geq 2$;

(ii) $[P_n^{(-2,\beta)}]'(1) = 0$ for all $n \geq 2$;

(iii) $[P_n^{(-2,\beta)}]''(x) = n(n-1)P_{n-2}^{(0,\beta+2)}(x)$ for all $n \geq 2$;

(iv) $\langle (1+x)_+^{\beta+2}H(1-x), P_m^{(0,\beta+2)}P_n^{(0,\beta+2)}\rangle = K_n(0,\beta+2)\delta_{mn}$, m and $n \geq 0$,

we obtain (4.50) from which the result follows. □

Case C.1.6: $\beta = -2$ **and** $\alpha \notin \{-3, -4, \cdots\}$. The differential equation (4.23) becomes

(4.51)
$$L[y](x) = (1 - x^2)y''(x) + ((-2 - \alpha) - \alpha x)y'(x) = -n(n + \alpha - 1)y(x).$$

The equation (4.51) is admissible and so has a unique monic PS$\{P_n^{(\alpha,-2)}(x)\}_{n=0}^{\infty}$ of solutions

$$P_n^{(\alpha,-2)}(x) = \binom{2n+\alpha-2}{n}^{-1} \sum_{k=0}^{n} \binom{n+\alpha}{k}\binom{n-2}{n-k}(x-1)^{n-k}(x+1)^k, \quad (n \geq 0).$$

The Sobolev weight equations corresponding to (4.51) are

(4.52) $$[(1-x^2)\sigma]' + [(\alpha+2) - \alpha x]\sigma = 0;$$

(4.53) $$(1-x^2)\tau' + [(\alpha+2) + \alpha x]\tau = 0;$$

(4.54) $$(1-x^2)\omega' - (\alpha+2)(1+x)\omega = 0.$$

The general solutions of (4.52), (4.53), and (4.54) are $\sigma = c_1[\alpha\delta(x+1) + 2\delta'(x+1)]$, $\tau = c_2\delta(x+1)$, $\omega = c_3(1-x)_+^{\alpha+2}H(1+x)$, where c_i $(i = 1, 2, 3)$ are arbitrary constants. As before we may consider only the following two parameter family of bilinear forms :

(4.55)
$$\begin{aligned}\phi(p,q) =& A[\beta p(-1)q(-1) - 2(p'(-1)q(-1) + p(-1)q'(-1))] \\ &+ Bp'(-1)q'(-1) + \langle (1-x)_+^{\alpha+2}H(1+x), p''(x)q''(x)\rangle.\end{aligned}$$

PROPOSITION 4.12. *The bilinear form* $\phi(\cdot,\cdot)$ *in* (4.55) *is*

(i) *quasi-definite if and only if* $\alpha A \neq 0$, $B - \frac{4}{\alpha}A \neq 0$;

(ii) *positive-definite if and only if* $\alpha A > 0$, $B - \frac{4}{\alpha}A > 0$.

In either case, the monic STPS or SOPS relative to $\phi(\cdot,\cdot)$ *is* $\{P_n^{(\alpha,-2)}(x)\}_{n=0}^{\infty}$ *and squared norms are given by*

(4.56)
$$\phi(P_n^{(\alpha,-2)}(x), P_n^{(\alpha,-2)}(x)) = \begin{cases} \alpha A & \text{if } n = 0 \\ B - \frac{4}{\alpha}A & \text{if } n = 1 \\ [n(n-1)]^2 K_{n-2}(\alpha+2, 0) & \text{if } n \geq 2 \end{cases}$$

where

$$K_n(\alpha+2,0) = \frac{2^{2n+\alpha+2}[n!\Gamma(n+\alpha+2)]^2}{\Gamma(2n+\alpha+4)\Gamma(2n+\alpha+3)} \quad (n \geq 0).$$

Proof. Proposition 4.4 shows that $\phi(P_m^{(\alpha,-2)}(x), P_n^{(\alpha,-2)}(x)) = 0$ for $m \neq n$. Since we have

(i) $P_n^{(\alpha,-2)}(-1) = 0$ for $n \geq 2$;
(ii) $[P_n^{(\alpha,-2)}]'(-1) = 0$ for $n \geq 2$;
(iii) $[P_n^{(\alpha,-2)}(x)]'' = n(n-1)P_{n-2}^{(\alpha+2,0)}(x)$ for $n \geq 2$;
(iv) $\langle (1-x)_+^{\alpha+2} H(1+x), P_m^{(\alpha+2,0)} P_n^{(\alpha+2,0)} \rangle = K_n(\alpha+2,0)\delta_{mn}$, m and $n \geq 0$,

we obtain (4.56) from which the result follows. □

Case C.2. Bessel type ($\ell_2(x) = x^2$) The differential equation is

$$x^2 y''(x) + (\alpha x + \beta) y'(x) = n(n+\alpha-1) y(x). \tag{4.57}$$

The Sobolev weight equations corresponding to (4.57) are

$$(x^2\sigma)' - (\alpha x + \beta)\sigma = 0; \tag{4.58}$$

$$x^2\tau' - (\alpha x + \beta)\tau = 0; \tag{4.59}$$

$$x^2\omega' - (\alpha x + \beta)\omega = 0. \tag{4.60}$$

The equation (4.58) has no quasi-definite moment functional solution σ if and only if $\alpha \in \{0, -1, \cdots\}$ or $\beta = 0$ and the equation (4.59) has no quasi-definite moment functional solution τ if and only if $\alpha + 2 \in \{0, -1, \cdots\}$ or $\beta = 0$, whereas the equation (4.60) has a quasi-definite moment functional solution ω if and only if $\alpha + 4 \notin \{0, -1, \cdots\}$ and $\beta \neq 0$. Hence, there arise two cases : $\alpha = -2$, $\beta \neq 0$ or $\alpha = -3$, $\beta \neq 0$.
For $\alpha = -2$, $\beta \neq 0$, the differential equation (4.57) has no polynomial solution of degree three.
For $\alpha = -3$, $\beta \neq 0$, the differential equation (4.57) has no polynomial solution of degree four.

Case C.3. Laguerre type ($\ell_2(x) = x$). The differential equation is

$$xy''(x) + (\alpha + 1 - x)y'(x) = -ny(x). \tag{4.61}$$

The Sobolev weight equations corresponding to (4.61) are

$$(x\sigma)' - (1 + \alpha - x)\sigma = 0; \tag{4.62}$$

$$x\tau' - (1 + \alpha - x)\tau = 0; \tag{4.63}$$

$$x\omega' - (2 + \alpha - x)\omega = 0. \tag{4.64}$$

The equation (4.62) has no quasi-definite moment functional solution σ if and only if $\alpha \in \{-1, -2, \dots\}$, and the equation (4.63) has no quasi-definite momentfunctional solution τ if and only if $\alpha \in \{-2, -3, \dots\}$, whereas the equation (4.64) has aquasi-definite moment functional solution ω if and only if $\alpha \notin \{-3, -4, \cdots\}$. Hence we have $\alpha = -2$ and then the above equations become, respectively,

$$L[y](x) = xy''(x) - (1 + x)y'(x) = -ny(x); \tag{4.65}$$

$$(x\sigma)' + (1 + x)\sigma = 0; \tag{4.66}$$

$$x\tau' + (1 + x)\tau = 0; \tag{4.67}$$

$$x\omega' + x\omega = 0. \tag{4.68}$$

The equation (4.65) is admissible and so has a unique monic PS$\{L_n^{(-2)}$ $(x)\}_{n=0}^{\infty}$ of solutions

$$L_n^{(-2)}(x) = (-1)^n n! \sum_{k=0}^{n} \binom{n-2}{n-k} \frac{(-x)^k}{k!} \qquad (n \geq 0).$$

The PS $\{L_n^{(-2)}(x)\}_{n=0}^{\infty}$ cannot be a TPS but it does form an STPS relative to the form of $\phi(\cdot, \cdot)$ in (4.10). The general solutions of (4.66), (4.67), and (4.68) are $\sigma = c_1[\delta(x) + \delta'(x)]$, $\tau = c_2\delta(x)$, and $\omega = c_3 H(x)e^{-x}$, where $c_i\,(i = 1, 2, 3)$ are arbitrary constants. As before we may consider only the following two parameter family of bilinear forms :

$$\begin{aligned}\phi(p, q) =& A[p(0)q(0) - (p'(0)q(0) + p(0)q'(0))] \\ &+ Bp'(0)q'(0) + \int_0^{\infty} p''(x)q''(x)e^{-x}\,dx.\end{aligned} \tag{4.69}$$

PROPOSITION 4.13. *The bilinear form $\phi(\cdot,\cdot)$ in (4.69) is*

(i) *quasi-definite if and only if $A \neq 0$, $B + 3A \neq 0$;*

(ii) *positive-definite if and only if $A > 0$, $B + 3A > 0$.*

In either case, the monic STPS or SOPS relative to $\phi(\cdot,\cdot)$ is $\{L_n^{(-2)}(x)\}_{n=0}^{\infty}$ and squared norms are given by

$$(4.70) \qquad \phi(L_n^{(-2)}(x), L_n^{(-2)}(x)) = \begin{cases} A & \text{if } n = 0 \\ B + 3A & \text{if } n = 1 \\ [n(n-1)(n-1)!]^2 & \text{if } n \geq 2. \end{cases}$$

Proof. Proposition 4.4 shows that $\phi(L_m^{(-2)}(x), L_n^{(-2)}(x)) = 0$ for $m \neq n$. Since we have

(i) $L_n^{(-2)}(0) = 0$ for all $n \geq 2$;

(ii) $[L_n^{(-2)}]'(0) = 0$ for all $n \geq 2$;

(iii) $[L_n^{(-2)}(x)]'' = n(n-1)L_{n-2}^{(0)}(x)$ for all $n \geq 2$;

(iv) $\langle H(x)e^{-x}, L_m^{(0)}L_n^{(0)}\rangle = (n!)^2\delta_{mn}$, m and $n \geq 0$,

we obtain (4.70) from which the result follows. □

Case C.4. Twisted Jacobi type ($\ell_2(x) = 1 + x^2$). The differential equation is

$$(4.71) \qquad (1 + x^2)y(x)'' + (bx + c)y(x)' = n(n + b - 1)y(x).$$

The Sobolev weight equations corresponding to (4.71) are

$$(4.72) \qquad [(1 + x^2)\sigma]' - (bx + c)\sigma = 0;$$

$$(4.73) \qquad (1 + x^2)\tau' - (bx + c)\tau = 0;$$

$$(4.74) \qquad (1 + x^2)\omega' - (bx + 2x + c)\omega = 0.$$

The equation (4.72) has no quasi-definite moment functional solution σ if and only if $b \in \{0, -1, -2, \dots\}$, and the equation (4.73) has no quasi-definite moment functional solution τ if and only if $b \in \{-2, -3, \dots\}$, whereas the equation (4.74) has a quasi-definite moment functional solution ω if and only if $b \notin \{-4, -5, \dots\}$. Hence, there arise two cases : $b = -2$ or $b = -3$. When $b = -3$, the differential equation (4.71) has no polynomial solution of degree four. When $b = -2$, the equation (4.71) has a PS of solutions only when $c = 0$ and then the above equations become

$$(4.75) \qquad L[y](x) = (1 + x^2)y''(x) - 2xy' = n(n - 3)y(x);$$

$$(4.76) \qquad [(1 + x^2)\sigma]' + 2x\sigma = 0;$$

$$(4.77) \qquad (1 + x^2)\tau' + 2x\tau = 0;$$

$$(4.78) \qquad (1 + x^2)\omega' = 0.$$

The equation (4.75) is not admissible but has monic polynomial solutions

$$\begin{aligned}\check{P}_n^{(-2,-2)}(x) &= \binom{2n-4}{n}^{-1} \sum_{k=0}^{n} \binom{n-2}{k}\binom{n-2}{n-k}(x-i)^{n-k}(x+i)^k \\ &= i^n P_n^{(-2,-2)}(-ix),\end{aligned}$$

for $n \neq 2, 3$ and $\check{P}_2(x) = x^2 + \gamma_1 x - 1$, $\check{P}_3(x) = x^3 + 3x + \gamma_2$, where γ_1, γ_2 are arbitrary constants. There are two linearly independent solutions σ, τ from (4.76), (4.77) respectively and one linearly independent solution ω from (4.78) such that

$$\sigma^{(1)} = -\sum_{n=0}^{\infty} \frac{(2n-1)}{(2n)!}\delta^{(2n)}(x), \quad \sigma^{(2)} = \sum_{n=1}^{\infty} \frac{n}{(2n+1)!}\delta^{(2n+1)}(x)$$

$$\tau^{(1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\delta^{(2n)}(x), \quad \tau^{(2)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\delta^{(2n+1)}(x)$$

and

$$\omega = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\delta^{(2n)}(x).$$

As before we may consider only the following four parameter family of bilinear forms :

(4.79)

$$\phi(p, q) = A\langle\sigma^{(1)}, pq\rangle + B\langle\sigma^{(2)}, pq\rangle + C\langle\tau^{(1)}, p'q'\rangle + D\langle\tau^{(2)}, p'q'\rangle + \langle\omega, p''q''\rangle.$$

PROPOSITION 4.14. *The bilinear form $\phi(\cdot,\cdot)$ in (4.79) is quasi-definite if and only if $A \neq 0$, $A - C \neq 0$, $4(C-A)(1-C) - (B+2D)^2 \neq 0$ and $30A^2 + 12A + 11B^2 \neq 0$. In this case, the monic STPS relative to $\phi(\cdot,\cdot)$ is $\{\check{P}_n^{(-2,-2)}(x)\}_{n=0}^{\infty}$ where $\check{P}_2^{(-2,-2)}(x) = x^2 + \gamma_1 x - 1$ and $\check{P}_3^{(-2,-2)}(x) = x^3 + 3x + \gamma_2$ with $\gamma_1 = \frac{B+2D}{C-A}$, $\gamma_2 = \frac{B}{A}$ and squared norms are given by*

(4.80)

$$\phi(\check{P}_n^{(-2,-2)}(x), \check{P}_n^{(-2,-2)}(x)) = \begin{cases} A & \text{if } n = 0 \\ C - A & \text{if } n = 1 \\ E & \text{if } n = 2 \\ F & \text{if } n = 3 \\ [n(n-1)]^2 \check{K}_{n-2}(0,0) & \text{if } n \geq 4 \end{cases}$$

where $E = (C-A)\gamma_1^2 - 2(B+2D)\gamma_1 + 4(1-C)$, $F = A(\gamma_2^2 - 30) - 12B\gamma_2 - 12$ and

$$\check{K}_n(0,0) = \int_{-1}^{1} [\check{P}_n^{(0,0)}(x)]^2\, dx = \frac{(-4)^n (n!)^4}{(2n)!(2n+1)!} \quad (n \geq 0).$$

Conversely, with any choice of γ_1, γ_2, $\{\check{P}_n^{(-2,-2)}(x)\}_{n=0}^{\infty}$ is an STPS relative to $\phi(\cdot,\cdot)$ if A, B, C, and D are such that

$$A \neq 0, \quad A - C \neq 0;$$

$$4(C-A)(1-C) - (B+2D)^2 \neq 0, \quad 30A^2 + 12A + 11B^2 \neq 0;$$

and

$$A\gamma_2 - B = 0, \quad (C-A)\gamma_1 - (B+2D) = 0.$$

Proof. The proof is similar to that of Proposition 4.10. □

Type D : σ, τ, ω are not quasi-definite. Recall that the bilinear form $\phi(\cdot,\cdot)$ in (4.10) may be quasi-definite even though all σ, τ and ω are not quasi-definite. We will now show that any STPS relative to such a bilinear form cannot satisfy a second-orderdifferential equation of the form (3.1).

THEOREM 4.15. *Let $\{P_n(x)\}_{n=0}^{\infty}$ be an STPS relative to aquasi-definite bilinear form $\phi(\cdot,\cdot)$ in (4.10). If σ,τ, ω are not quasi- definite, then $\{P_n(x)\}_{n=0}^{\infty}$ can not satisfy a second-orderdifferential equation of the form (3.1).*

Proof. Assume that $\{P_n(x)\}_{n=0}^{\infty}$ satisfies the differential equation(3.1) but σ, τ, and ω are not quasi-definite. Then, $\{P_n'(x)\}_{n=1}^{\infty}$ and $\{P_n''(x)\}_{n=2}^{\infty}$ satisfy the differential equation (4.9) for $k = 1, 2$ respectively and σ, τ, and ω satisfy the Sobolev weight equations (4.3) for $k = 0, 1, 2$ respectively. Since σ, τ, and ω are not quasi-definite, Theorem 3.5 and Theorem 3.6 imply

$$\langle\sigma, P_n^2\rangle = \langle\tau, (P_n')^2\rangle = \langle\omega, (P_n'')^2\rangle = 0$$

for all n large enough so that

$$\phi(P_n, P_n) = 0$$

for all n large enough, which contradicts the fact that $\{P_n(x)\}_{n=0}^{\infty}$ is an STPS relative to $\phi(\cdot,\cdot)$. □

REMARK 4.3. Theorem 4.15 can be generalized to the case of the symmetric bilinear form $\phi(\cdot,\cdot)$ in (2.7). In other words, any STPS relative to a symmetric bilinear form in (2.7) with no quasi-definite moment functional $\sigma_{(k)}$ for $k = 0, 1, \cdots N$ cannot satisfy a second order differential equation of the form (3.1).

In summary, we have shown that there are, up to a real linear change of variable, eighteen distinct PS's satisfying the differential equation (3.1) which are orthogonal relative to a bilinear form $\phi(\cdot,\cdot)$ in (4.10).

ACKNOWLEDGEMENTS. This work is partially supported by Global Analysis Research Center and Ministry of Education (BSRI 1420).

References

1. S. Bochner, *Über Sturm-Liouvillesche Polynomsysteme*, Math. Z. **89** (1929), 730-736.
2. R. P. Boas, *The Stieltjes moment problem for functions of bounded variation*, Bull. Amer. Math. Soc. **45** (1939), 399-404.

3. A. J. Duran, *The Stieltjes moment problem for rapidly decreasing functions*, Proc. Amer. Math. Soc. **107** (1989), 731-741.
4. A. J. Duran, *A generalization of Favard's theorem for polynomials satisfying a recurrence relation*, J. Approx. Th., **74** (1993), 83-109.
5. L. Hörmander, *The analysis of linear partial differential operators I*, Springer-Verlag, New-York, 1935.
6. H. L. Krall and I. M. Sheffer, *Differential equations of infinite order for orthogonal polynomials*, Ann. Mat. Pura Appl. **74** (1966), 135-172.
7. K. H. Kwon and L. L Littlejohn, *Classification of classical orthogonal polynomials*, submitted for publication.
8. K. H. Kwon and L. L. Littlejohn, *Classification of Sobolov orthogonal polynomials satisfying second order differential equations*, RIM-GARC Preprint Series 93-26 (1993).
9. K. H. Kwon and L. L. Littlejohn, *Sobolev orthogonal polynomials and second-order differential equations*, submitted for publication.
10. F. Marcellán, M. Alfaro, and M. L. Rezola, *Orthogonal polynomials on Sobolev spaces : old and new directions*, J. Comp. Appl. Math., **48** (1993), 113-131.
11. P. Maroni, *Une theorie algebrique des polynomes orthogonaux. Application aux polynomes orthogonaux semi-classiques*, in Orthgonal Polynomials and their applications, IMACS Annals on Computing and Applied Mathematics, vol. 9, (L. Gori, C. Brezinski, A. Ronveaux), J.C. Baltzer AG, Basel, Switzerland, 1991, pp. 95-130.
12. R. D. Morton and A. M. Krall, *Distributional weight functions for orthogonal polynomials*, SIAM J. Math. Anal. **9(3)** (1978), 604-626.
13. A. Ronveaux, *Sobolev inner products and orthogonal polynomials of Sobolev type*, Numer Algor. **3** (1992), 393-400.

K. H. Kwon and D. W. Lee
Department of Mathematics, KAIST, Taejon 305-701, Korea
E-mail: khkwon@jacobi.kaist.ac.kr

L. L. Littlejohn
Department of Mathematics and Statistics, Utah State University, Logan Utah 84322-3900, U.S.A.
E-mail: lance@sunfs.math.usu.edu