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A COHESIVE MATRIX IN A CONJECTURE ON PERMANENTS

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1. Introduction and Preliminaries

Let Ω_n be the polyhedron of $n \times n$ doubly stochastic matrices, that is, nonnegative matrices whose row and column sums are all equal to 1. The *permanent* of a $n \times n$ matrix $A = [a_{ij}]$ is defined by

$$\mathrm{per}(A) = \sum_{\sigma} a_{1\sigma(1)} \cdots a_{n\sigma(n)}$$

where σ runs over all permutations of $\{1, 2, \dots, n\}$.

Let $D = [d_{ij}]$ be an $n \times n$ $(0,1)$-matrix, and let

$$\Omega(D) = \{X = [x_{ij}] \in \Omega_n \mid x_{ij} = 0 \text{ whenever } d_{ij} = 0\}.$$

Then $\Omega(D)$ is a face of Ω_n, and since it is compact, $\Omega(D)$ contains a *minimizing matrix* A such that per(A)$\leq$per(X) for all $X \in \Omega(D)$.

Let R_n denote the $n \times n$ $(0,1)$-matrix with zero trace, and all off-diagonal entries equal to 1, and E_{ij} denote the $n \times n$ matrix whose (i,j) entry is 1, and whose other entries are all zero. Let $C_n = R_n + E_{11}$ and J_n be the $n \times n$ matrix with all entries equal to 1.

Brualdi [1] defined an $n \times n$ $(0,1)$-matrix D to be *cohesive* if there is a matrix $A = [a_{ij}]$ in the interior of $\Omega(D)$(that is, $a_{ij} \neq 0$ whenever $d_{ij} = 1$) for which

$$\mathrm{per}(A) = \min\{\mathrm{per}(X) \mid X \in \Omega(D)\},$$

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and he defined an $n \times n$ $(0,1)$-matrix D to be *barycentric* if

$$\mathrm{per}(B(D)) = \min \{\mathrm{per}(X) \mid X \in \Omega(D)\}$$

where the barycenter $b(D)$ of $\Omega(D)$ is given by

$$b(D) = \frac{1}{\mathrm{per}(D)} \sum_{P \leq D} P,$$

where the summation extends over the set of all permutation matrices P with $P \leq D$ and per(D) is their number.

It is true that a (0,1)-matrix can be cohesive without being barycentric. Such an example was given in [7] by Song. In [1], Brualdi conjectured that C_n(defined above) would be a likely candidate for such an interesting example. Minc [6] gave a local minimizing matrix $X_n(\alpha)$ on $\Omega(C_n)$;

$$(1) \qquad X_n(\alpha) = \begin{pmatrix} \beta & \alpha & \alpha & \dots & \alpha & \alpha \\ \alpha & 0 & \gamma & \dots & \gamma & \gamma \\ \alpha & \gamma & 0 & & \gamma & \gamma \\ \vdots & \vdots & & \ddots & & \vdots \\ \alpha & \gamma & \gamma & \dots & 0 & \gamma \\ \alpha & \gamma & \gamma & \dots & \gamma & 0 \end{pmatrix}$$

where $\alpha = \frac{\mathrm{per}(R_{n-1})}{d}$, $\beta = \frac{(n-2)\mathrm{per}(R_{n-2})}{d}$ and $\gamma = \frac{\mathrm{per}(C_n)}{d}$ with $d = \mathrm{per}(C_n) - \mathrm{per}(C_{n-1})$. And Song [8] proved directly that C_n is never barycentric for $n \geq 3$. But, no one did determine the minimum permaent and minimizing matrix on $\Omega(C_n)$ for $n \geq 4$. Moreover, it was not proved that C_n is cohesive for $n \geq 4$.

In this paper, we prove that C_4 is cohesive. The general case remains open.

Let A be an $n \times n$ nonnegative matrix. If column k of A contains exactly two nonzero entries, say in rows i and j, then the $(n-1) \times (n-1)$ matrix $C(A)$ obtained from A by replacing row i with the sum of rows i and j and deleting row j and column k is called a *contraction* of A. If A has a row with exactly two nonzero entries, then $C(A^t)^t$ is also a contraction of A, where A^t is the transpose of A.

LEMMA 1 ([3]). *Suppose $A \in \Omega_n$ is fully indecomposable and has a column (row) with exactly two positive entries. Then $\overline{C(\bar{A})}$ is fully indecomposable and $(n-1) \times (n-1)$ doubly stochastic, and*

$$2per(A) \geq 2per(\bar{A}) = per(C(\bar{A})) \geq per(\overline{C(\bar{A})}),$$

where $\bar{A}$ $(\overline{C(\bar{A})})$ is a minimizing matrix on the face $\Omega(A)$(or $\Omega(C(\bar{A}))$, respectively) of Ω_n.■

The following Lemma is a known result (See [3] or [7]).

LEMMA 2. *If $A = [a_{ij}]$ is a minimizing matrix on $\Omega(C_n)$ then $per(A(i \mid j)) \geq per(A)$ for $a_{ij} = 0$ and $c_{ij} = 1$.*■

2. The cohesiveness of C_4

Egorychev [2] proved that $\frac{1}{n}J_n$ is the unique minimizing matrix on Ω_n. After that, determining the minimizing matrix and minimum permanent on $\Omega(R_n)$ is one of the famous problems on permanents. London and Minc [4] proved that $\frac{1}{3}R_4$ is the unique minimizing matrix on $\Omega(R_4)$. But the general case on $\Omega(R_n)$ remains open. We use this result in the proof of the cohesivness of C_4.

THEOREM 3 ([4]). *For any $A \in \Omega(R_4)$, $per(A) \geq per(\frac{1}{3}R_4) = \frac{1}{9}$. Equality holds only if $A = \frac{1}{3}R_4$.*■

THEOREM 4. *The matrix C_4 is cohesive.*

The proof follows from the Lemmas 5, 6 and 7.

LEMMA 5. *If $A = [a_{ij}]$ is a minimizing matrix on $\Omega(C_4)$, then a_{11} is not zero.*

Proof. Suppose that $a_{11} = 0$. Then A is contained in $\Omega(R_4)$. Thus

$$\text{per}(A) \geq \text{per}(\frac{1}{3}R_4) = \frac{1}{9} \tag{2}$$

by Theorem 3. Consider a likely candidate

$$X_4 = \frac{1}{8}\begin{pmatrix} 2 & 2 & 2 & 2 \\ 2 & 0 & 3 & 3 \\ 2 & 3 & 0 & 3 \\ 2 & 3 & 3 & 0 \end{pmatrix}$$

in (1) for a minimizing matrix on $\Omega(C_4)$. Then

$$\text{per}(X_4) = \frac{27}{256}, \tag{3}$$

which is less than $\frac{1}{9}$. Hence A with $a_{11} = 0$ can not be a minimizing matrix by (2) and (3).■

LEMMA 6. *If $A = [a_{ij}]$ is a minimizing matrix on $\Omega(C_4)$, then a_{1j} and a_{j1} are not zero for $j = 2, 3$ and 4.*

Proof. Assume that $a_{12} = 0$. Then a_{23} and a_{24} cannot be zero since A is fully indecomposable by Lemma 2. That is, the second column of A has only two nonzero entries. Now, consider the contraction $C(A)$ of A on the second column;

$$C(A) = \begin{pmatrix} a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} \\ a_{31} + a_{41} & a_{43} & a_{34} \end{pmatrix}$$

Then $C(A)$ is contained in $\Omega(J_3)$. Thus

$$\text{per}(C(A)) \geq \text{per}(\frac{1}{3}J_3) = \frac{2}{9} \tag{4}$$

by the van der Waerden-Egorychev Theorem [2]. And Lemma 2 implies that

$$2\text{per}(A) \geq \text{per}(C(A)) \tag{5}$$

Since X_4 in (1) has $\frac{27}{256}$ as its permanent from (3), we have

$$\text{per}(A) \geq \frac{1}{9} > \frac{27}{256} = \text{per}(X_4)$$

by (4) and (5). Therefore A with $a_{12} = 0$ is not a minimizing matrix on $\Omega(C_4)$.

By the similar proof, we can show that a_{1j} is not zero for $j = 3, 4$. If a_{j1} is zero for $j = 2, 3$ and 4, then we can show that A is not a minimizing matrix on $\Omega(_4)$ by the use of contraction on the jth row of A. ■

LEMMA 7. *If $A = [a_{ij}]$ is a minimizing matrix on $\Omega(C_4)$, then a_{ij} is not zero for $i, j = 2, 3$ and 4 with $i \neq j$.*

Proof. Assume that $a_{23} = 0$. Then a_{13} and a_{43} cannot be zero since A is fully indecomposable by Lemma 2. That is, the third column of A has only two nonzero entries. Thus the contraction $C(A)$ of A on the third column has the form

$$C(A) = \begin{pmatrix} a_{11} + a_{41} & a_{12} + a_{42} & a_{14} \\ a_{21} & 0 & a_{24} \\ a_{31} & a_{32} & a_{34} \end{pmatrix}$$

Since $C(A)$ is contained in $\Omega(J_3 - E_{22})$, per$(C(A))$ is greater than or equals the minimum permanent on $\Omega(J_3 - E_{22})$. But the minimum permanent on $\Omega(J_3 - E_{22})$ is $\frac{1}{4}$ by Theorem 1 in [1] because the permanent is invariant under the exchange of rows or columns. That is,

$$\text{per}(C(A)) \geq \frac{1}{4}. \tag{6}$$

Since X_4 in (1) has $\frac{27}{256}$ as its permanent from (3), we have

$$\text{per}(A) \geq \frac{1}{2}\text{per}(C(A)) \geq \frac{1}{8} > \frac{27}{256} = \text{per}(X_4)$$

by (5) and (6). Therefore A with $a_{23} = 0$ is not a minimizing matrix on $\Omega(C_4)$. By the similar proof, we can show that a_{ij} is not zero for $i, j = 2, 3$ and 4 with $i \neq j$. ■

Proof of Theorem 4. Assume that $A = [a_{ij}]$ is a minimizing matrix on $\Omega(C_4)$. Then Lemmas 5 and 6 imply that a_{i1} and a_{1i} are not zero for $i = 1, 2, 3$ and 4. Lemma 7 implies that a_{ij} is not zero for $i, j = 2, 3$

and 4 with $i \neq j$. Hence the minimizing matrix on $\Omega(C_4)$ is in the interior of the face $\Omega(C_4)$. That is, C_4 is cohesive, as required. ■

As a concluding remark, we propose a problem which is related to the conjecture of R. A. Brualdi in [1].

Problem Does the assumption that $\frac{1}{n-1}R_n$ is a minimizing matrix on $\Omega(R_n)$ imply the cohesiveness of C_n for $n \geq 4$?

In this paper, we see that the answer for this problem is yes for $n = 4$. Here we have a partial result on this problem.

PROPOSITION 8. *Let $A_n = [a_{ij}]$ be a minimizing matrix on $\Omega(C_n)$. If $\frac{1}{n-1}R_n$ is a minimizing matrix on $\Omega(R_n)$, then the entry a_{11} in A_n is not zero.*

Proof. Assume that a_{11} in A_n is zero. Then A_n is contained in the face $\Omega(R_n)$. Since $\frac{1}{n-1}R_n$ is a minimizing matrix on $\Omega(R_n)$, $\frac{1}{n-1}R_n$ is a minimizing matrix on $\Omega(C_n)$. Hence we have $\text{per}(\frac{1}{n-1}R_n) \leq \text{per}(\frac{1}{n-1}R_n(1|1))$ by Lemma 2. But

$$\text{per}(A_n) = \text{per}(\frac{1}{n-1}R_n) = (\frac{1}{n-1})^n \text{per}(R_n)$$

and

$$\text{per}(A_n(1 \mid 1)) = \text{per}(\frac{1}{n-1}R_n(1 \mid 1)) = (\frac{1}{n-1})^{n-1} \text{per}(R_{n-1}).$$

Thus we have that $\text{per}(R_n) \leq (n-1)\text{per}(R_{n-1})$. However, for $n \geq 4$, we have $\text{per}(R_n) = (n-1)[\text{per}(R_{n-1}) + \text{per}(R_{n-2})]$(see [5] Page 44), which is greater than $(n-1)\text{per}(R_{n-1})$. Hence we have a contradiction. This implies that a_{11} in A_n is not zero. ■

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