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A REDUCIBLE CASE OF DOUBLE HYPERGEOMETRIC SERIES INVOLVING THE RIEMANN ζ–FUNCTION

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1. Introduction

Using the Pochhammer symbol $(\lambda)_n$ given by

$$(1.1)\quad (\lambda)_n = \begin{cases} 1, & \text{if } n = 0, \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & \text{if } n \in N = \{1, 2, 3, \dots\}, \end{cases}$$

we define a general double hypergeometric series by [3, p. 27]

$$(1.2)\quad \begin{aligned} &F^{p:r;u}_{q:s;v}\left[\begin{matrix} \alpha_1,\dots,\alpha_p: & \gamma_1,\dots,\gamma_r; & \lambda_1,\dots,\lambda_u; \\ \beta_1,\dots,\beta_q: & \delta_1,\dots,\delta_s; & \mu_1,\dots,\mu_v; \end{matrix} x, y\right] \\ &= \sum_{l,m=0}^{\infty} \frac{\prod_{j=1}^{p}(\alpha_j)_{l+m}\prod_{j=1}^{r}(\gamma_j)_l\prod_{j=1}^{u}(\lambda_j)_m}{\prod_{j=1}^{q}(\beta_j)_{l+m}\prod_{j=1}^{s}(\delta_j)_l\prod_{j=1}^{v}(\mu_j)_m}\frac{x^l}{l!}\frac{y^m}{m!} \end{aligned}$$

provided that the double series converges.

Srivastava and Karlsson [3, pp. 28-32] have recorded a large number of instances in which the double hypergeometric series (1.2) reduces to simpler functions including, for example, a hypergeometric function of a single variable [3, p. 299]. More recently, Srivastava and Miller [4] presented a reducible case of a double hypergeometric series involving Catalan's constant and Riemann's ζ-function. The object of present paper is to point out a particularly simple reducible case of (1.2) when

$$(1.3)\quad p = q = r - 1 = s = u - 1 = v = 1 \qquad \text{and} \qquad x = y = 1,$$

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and the resulting parameters are appropriately specialized. Indeed, in terms of the Riemann ζ-function (see, for example, Whittaker and Watson [5], chapter 13):

$$(1.4) \qquad \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \operatorname{Re}(s) > 1,$$

we shall show that

$$(1.5) \qquad F_{1:1;1}^{1:2;2}\left[\begin{matrix} 1: & \frac{1}{2}, \frac{1}{2}; & \frac{1}{2}, \frac{1}{2}; \\ 2: & \frac{3}{2}; & \frac{3}{2}; \end{matrix} 1, 1\right] = \frac{\pi^2}{2} \ln 2 - \frac{7}{4}\zeta(3),$$

where $\zeta(3) \cong 1.202\,056\,903\,159\,594\,285\ldots$.

2. Derivation of the Reduction Formula (1.5)

Denote, for convenience, the left-hand side of the reduction formula (1.5) by Υ. Since [2, p. 17, Ex. 18]

$$(2.1) \qquad \arcsin x = xF(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2),$$

in terms of the Gaussian hypergeometric function, we readily have

$$(2.2) \qquad \Upsilon = 2\int_0^1 \frac{(\arcsin x)^2}{x} dx.$$

Indeed, using the Cauchy product of two infinite series, we have

$$(2.3) \qquad \begin{aligned} \frac{(\arcsin x)^2}{x} &= x \sum_{l=0}^{\infty} \sum_{m=0}^{l} \frac{(\frac{1}{2})_m (\frac{1}{2})_m}{(\frac{3}{2})_m : m!} \frac{(\frac{1}{2})_{l-m} (\frac{1}{2})_{l-m}}{(\frac{3}{2})_{l-m} : (l-m)!} x^{2l} \\ &= \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m (\frac{1}{2})_m (\frac{1}{2})_l (\frac{1}{2})_l}{(\frac{3}{2})_m (\frac{3}{2})_l} \frac{x^{2l+2m+1}}{l! : m!}, \end{aligned}$$

where we use the index formula [2, p. 57, Eq. (2)] for the second equality of (2.3).

Integrating both sides of the equation (2.3) from $x = 0$ to $x = 1$, multiplying both sides of the resulting equation by 2, and noting that $(1)_{l+m}/(2)_{l+m} = 1/(l+m+1)$, we obtain the desired formula (2.2).

Setting $x = \sin t$ in (2.2), we have

$$\Upsilon = 2\int_0^{\frac{\pi}{2}} t^2 \cot t dt. \tag{2.4}$$

Integrating by parts, we find from (E2.4) that

$$\Upsilon = \frac{\pi^2}{2}\ln 2 - 4\int_0^{\frac{\pi}{2}} t \ln \sin t \, dt. \tag{2.5}$$

Now, using the elementary trigonometric expansion [1, p. 356]:

$$\ln(2\sin t) = -\sum_{n=1}^{\infty} \frac{\cos 2nt}{n} \qquad (0 < t < \pi), \tag{2.6}$$

and inverting the order of integration and summation, we therefore obtain

$$\Upsilon = \frac{\pi^2}{2}\ln 2 + 4\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\frac{\pi}{2}} t \cos 2nt \, dt. \tag{2.7}$$

Integrating this last integral by parts, we have

$$\begin{aligned} \Upsilon &= \frac{\pi^2}{2}\ln 2 + \sum_{n=1}^{\infty}\frac{1}{n^3}\left[(-1)^n - 1\right] \\ &= \frac{\pi^2}{2}\ln 2 - 2\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}. \end{aligned} \tag{2.8}$$

Recall the known result [5, p. 271]:

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^s} = \left(1 - 2^{-s}\right)\zeta(s), \qquad \mathrm{Re}(s) > 1. \tag{2.9}$$

Finally, applying the special case $s = 3$ of (2.9) and replacing the right-hand side of the second equation in (2.8) by the resulting formula for $\zeta(3)$, we obtain the reduction formula (1.5).

3. Concluding Remarks

Our derivation of the reduction formula (1.5) provides yet another interesting illustration of the familiar method of obtaining transformation and reduction formulas for multiple hypergeometric series by evaluating definite integrals in two different ways. For numerous other illustrations of this rather fruitful method, one should refer to Srivastava and Karlsson [3, pp. 325-331].

It may also be remarked that, as an interesting by-product of our analysis presented in the preceding section in detail, we find that

$$\int_0^1 \frac{(\arcsin x)^2}{x}\,dx = \frac{\pi^2}{4}\ln 2 - \frac{7}{8}\zeta(3). \tag{3.1}$$

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