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MEAN ERGODIC THEOREM AND MULTIPLICATIVE COCYCLES

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1. Introduction

Let $(X, \mathcal{B}, \mu)$ be a probability space. Then we say $\tau : X \to X$ is a measure-preserving transformation if $\mu(\tau^{-1}E) = \mu(E)$. and we call it an ergodic transformation if $\mu(\tau^{-1}E \triangle E) = 0$ for a measurable subset E implies $\mu(E) = 0$. An equivalent definition is that constant functions are the only τ-invariant functions.

Let G be a compact abelian group with its normalized Haar measure and Γ a countably infinite dense subgroup. Let $\widehat{G}$ denote the dual group consisting of characters of G. Recall that $\widehat{G}$ is discrete and that the characters form an orthonormal basis for the Hilbert space $L^2(G)$. For example, let $\mathbb{R}$ be the additive group of real numbers, $\mathbb{Z}$ its subgroup of integers. Then the quotient group $\mathbb{R}/\mathbb{Z}$ is just the unit circle $\mathbb{T}$ identified with the half open interval $[0, 1)$. Its dual group is $\mathbb{Z}$. Let τ_g be the translation in a compact abelian group G by an element g. It preserves the Haar measure on G. It is ergodic if and only if the subgroup $\{ng : n \in \mathbb{Z}\}$ is dense in G. If G is the unit circle $[0, 1)$, then g generates a dense subgroup if and only if g is an irrational number.

Multiplicative cocycles were first studied by Helson to investigate the Wiener type or Beurling type invariant subspaces on compact abelian groups. Here is a formal definition:

DEFINITION. Let G be a compact abelian group and Γ a dense subgroup. A function A on $\Gamma \times G$ is called a multiplicative cocycle defined on Γ if it satisfies the following:

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(i) $|A(\gamma, x)| = 1$ almost everywhere with respect to μ for every $\gamma \in \Gamma$.

(ii) $A_\gamma \equiv A(\gamma, \cdot)$ is a measurable function on G for every γ in Γ.

(iii) $A(\gamma_1 + \gamma_2, x) = A(\gamma_1, x)A(\gamma_2, x - \gamma_1)$ a. e. with respect to μ for every γ_1, γ_2 in Γ.

From now on, by cocycles we simply mean multiplicative cocycles if there is no ambiguity. For the applications of cocycles arising from irrational rotations on the circle, see [2],[3].

A continuous unitary representation of a compact group G on a Hilbert space $\mathcal{H}$ is a group homomorphism $g \mapsto U_g$ from G into the group of unitary operators $\mathcal{U}(\mathcal{H})$ such that the map $g \mapsto U_g(h)$ is continuous from G into $\mathcal{H}$ for each fixed $h \in \mathcal{H}$. Then for each vector $h \in \mathcal{H}$ there is a unique positive Borel measure μ_h on $\widehat{G}$ such that

$$(U_g h, h) = \int_{\widehat{G}} \chi(g)\, d\mu_h(\chi)$$

where $(\ ,\ )$ denotes the inner product of $\mathcal{H}$. The proof follows from Bochner's theorem, since the map $g \mapsto (U_g h, h)$ is a positive definite function on G. In fact, the measures μ_h are obtained from a single spectral measure P on $\widehat{G}$ satisfying $(P(E)h, h) = \mu_h(E)$ for measurable subsets $E \subset \widehat{G}$, so that

$$U_g = \int_{\widehat{G}} \chi(g)\, dP(\chi).$$

For details on unitary representations, see [5].

PROPOSITION 1. *Let G be a compact abelian group with the normalized Haar measure μ. For a dense subgroup Γ we are given a cocycle A. Define $U_\gamma : L^2(G, \mu) \to L^2(G, \mu)$ by the formula*

$$(U_\gamma f)(x) = A(\gamma, x) f(x - \gamma)$$

where $x \in G$, $f \in L^2(G)$ for every $\gamma \in \Gamma$. Then $\{U_\gamma\}_{\gamma \in \Gamma}$ is a (not necessarily continuous) unitary representation of Γ

REMARK. Sometimes Γ is endowed with the discrete topology so that the mapping $\gamma \to U_\gamma$, $f \in L^2(G)$ is automatically continuous from Γ into $L^2(G)$.

Proof. It is obvious that $||U_\gamma f||_2 = ||f||_2$ since $|A(\gamma, x)| = 1$ a.e. with respect to μ for every $\gamma \in \Gamma$. Now let us show that $U_{\gamma_1+\gamma_2} = U_{\gamma_1} U_{\gamma_2}$ for $\gamma_1, \gamma_2 \in \Gamma$. Take $x \in G$, $f \in L^2(G)$. Then we have

$$\begin{aligned}(U_{\gamma_1} U_{\gamma_2} f)(x) &= U_{\gamma_1}(A(\gamma_2, x) f(x - \gamma_2)) \\ &= A(\gamma_1, x) A(\gamma_2, x - \gamma_1) f(x - (\gamma_1 + \gamma_2)) \\ &= (U_{\gamma_1+\gamma_2} f)(x). \qquad \square\end{aligned}$$

DEFINITION. Let q be a measurable function on G and $|q(x)| = 1$ a.e. with respect to μ. Define $B(\gamma, x) = \overline{q(x)} q(x - \gamma)$. Then $B : \Gamma \times G \to \mathbb{T}$ satisfies

$$\begin{aligned}B(\gamma_1 + \gamma_2, x) &= \overline{q(x)} q(x - \gamma_1 - \gamma_2) \\ &= (\overline{q(x)} q(x - \gamma_1))(\overline{q(x - \gamma_1)} q(x - \gamma_1 - \gamma_2)) \\ &= B(\gamma_1, x) B(g_2, x - \gamma_1).\end{aligned}$$

Hence B is a cocycle. We call it a *multiplicative coboundary*, or a coboundary if there is no danger of ambiguity. Sometimes Γ is generated by one element γ_0. Then the relation $B(\gamma_0, x) = \overline{q(x)} q(x - \gamma_0)$ defines a coboundary on Γ uniquely and B satisfies $B(n\gamma_0, x) = \overline{q(x)} q(x - n\gamma_0)$. In general, if a function $f(x)$ of modulus 1 a.e. is of the form $f(x) = \overline{q(x)} q(x - n\gamma_0)$, then we also call it a coboundary.

For irrational rotations, coboundaries are related with uniform distribution of integral multiples of irrational numbers as pointed out in [7]: For an irrational number $\theta \in \mathbb{T}$ and an interval $I \subset \mathbb{T}$, we define

$$S_n(x) = \sum_{j=0}^{n-1} \chi_I(x - j\theta) = \operatorname{card}\{j : 0 \le j < n, x - j\theta \in I\}$$

where χ_I is the characteristic function of I. Then Weyl-Kronecker theorem says that for every x

$$\lim_{n\to\infty} \frac{1}{n} S_n(x) = m(I)$$

where m is the Lebesgue measure on $\mathbb{T}$. Now let $x_j \in \{0,1\}$ be such that $x_j \equiv S_j(0) \pmod 2$. Veech[7] proved that for every irrational $\theta \in \mathbb{T}$ there exists an interval I, depending on θ, for which $\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} x_j$ does not exist. Let $\theta = [a_1, a_2, ..., a_k, ...]$ be the continued fraction expansion of irrational $0 < \theta < 1$, where $a_1, a_2, ..., a_k, ...$ are called *partial quotients*, and $m_k/n_k = [a_1, a_2, ..., a_k]$, $(m_k, n_k) = 1$, are called *convergents*. They satisfy $|\theta - m_k/n_k| < 1/(2n_k{}^2)$ for every $k \geq 1$. The irrational numbers with bounded partial quotients form a set of measure zero. The limit, not necessarily equal to $1/2$, exists for every interval $I \subset \mathbb{T}$ if and only if θ has bounded partial quotients in its continued fraction expansion. If $\exp(\pi i \chi_I)$ is a coboundary, then the limit is not equal to $1/2$. For θ with bounded partial quotients, $\exp(\pi i \chi_I)$ is a multiple of a coboundary if and only if $m(I) \in \mathbb{Z}\cdot\theta + \mathbb{Z}$. If θ has unbounded partial quotients, then $\exp(\pi i \chi_I)$ is a multiple of a coboundary for uncountably many values of the length $m(I)$. For the application of coboundaries for uniform distribution of the orbits under general measure preserving transformations, see [1].

In this article we show that a multiplicative cocycle giving a continuous unitary representation of a countably dense subgroup of a compact abelian group is a multiplicative coboundary.

2. Main Result

PROPOSITION 2. *If $A(\gamma, x)$ is a coboundary, then the corresponding unitary representation $\{U_\gamma\}$ is unitarily equivalent to $\{T_\gamma\}$ where $T_\gamma : G \to G$ is the translation by γ.*

Proof. Since $A(\gamma, x) = \overline{q(x)}q(x-\gamma)$ for some q, we have

$$(U_\gamma f)(x) = \overline{q(x)}q(x-\gamma)f(x-\gamma) = (\overline{q}T_\gamma(qf))(x).$$

So $U_\gamma f = \overline{q}T_\gamma(qf)$, $M_q U_\gamma = T_\gamma M_q$, where M_q means the unitary operators defined by multiplication by q. What we have here is the following diagram:

$$\begin{array}{ccc} L^2(G) & \xrightarrow{U_g} & L^2(G) \\ {\scriptstyle M_q}\downarrow & & {\scriptstyle M_q}\downarrow \\ L^2(G) & \xrightarrow{T_\gamma} & L^2(G) \end{array} \qquad \square$$

Let (X, m) be a σ-finite measure space, $T : X \to X$ a measure-preserving transformation, and $f \in L^2(X, m)$. Then the classical Mean Ergodic Theorem due to von Neumann states that there is $\overline{f} \in L^2(X, m)$ for which $\frac{1}{n}\sum_{k=0}^{\infty} f \circ T^k$ converges to $\overline{f}$ in L^2. In general, if U is a contraction on a Hilbert space $\mathcal{H}$, i.e., $||Uf|| \leq ||f||$ for $f \in \mathcal{H}$, and if $\mathcal{M} = \{h \in \mathcal{H} : Uf = f\}$ and $P : \mathcal{H} \to \mathcal{H}$ the projection of $\mathcal{H}$ onto $\mathcal{M}$, then $\frac{1}{n}\sum_{k=0}^{n-1} U^k f$ converges to Pf in $\mathcal{H}$. For the proofs, see P.23, [6].

The following result might be called an integral version of von Neumann's Mean Ergodic Theorem.

PROPOSITION 3. *Let G be a compact abelian group and $\{U_g\}_{g\in G}$ a continuous unitary representation of G in a Hilbert space $\mathcal{H}$. Let P be the self-adjoint orthogonal projection onto the subspace*

$$\mathcal{H}_1 = \{h \in \mathcal{H} : U_g h = h \text{ for every } g \in G\}.$$

Then P satisfies the relation

$$\int_G U_g h \, d\mu(g) = Ph$$

for every $h \in \mathcal{H}$, where $d\mu$ is the normalized Haar measure on G.

Proof. Since $\{U_g\}$ is a continuous unitary representation, we can find a spectral measure on the dual group $\widehat{G}$, which is discrete and satisfies the following:

$$U_g = \sum_{\chi \in \widehat{G}} \chi(g) P_\chi$$

where $\{P_\chi\}$ is a family of mutually orthogonal self-adjoint projections in $L^2(G)$ such that $\sum_\chi P_\chi = 1$. Hence we have

$$\int_G U_g h \, d\mu(g) = \sum_{\chi \in \widehat{G}} \left\{ \int_G \chi(g) d\mu(g) \right\} P_\chi h.$$

But $\int_G \chi(g) d\mu(g) = 0$ if and only if $\chi \neq 1$. Thus

$$\int_G U_g h \, d\mu(g) = P_1 h$$

where P_1 is the orthogonal projection corresponding to $\chi \equiv 1$.

Now we show that $\mathcal{H}_1 = \{h \in \mathcal{H} : P_1 h = h\}$, that is, $P = P_1$. If $h \in \mathcal{H}_1$, then $U_g h = h$ for all $g \in G$, hence $h = P_1 h$. If $P_1 h = h$, then $P_\chi h = 0$ for $\chi \neq 1$. So we have

$$U_g h = \sum_{\chi \in \widehat{G}} \chi(g) P_\chi h = P_1 h = h. \qquad \square$$

In [4] it is shown that $\{U_\lambda\}_{\lambda \in \mathbb{R}}$ is a unitary representation of $\mathbb{R}$ in $L^2(\mathbb{R})$ given by a cocycle $A : \mathbb{R} \times \mathbb{R} \to \mathbb{T}$ as in Proposition 1. Then $\{U_\lambda\}_{\lambda \in \mathbb{R}}$ is a continuous unitary representation of $\mathbb{R}$ if and only if A is a coboundary. The proof uses Weyl commutation relation and spectral theory. In the following we prove a similar result for a compact abelian group using the Mean Ergodic Theorem. This illustrates an aspect of the invariant subspace method that is used in [1].

THEOREM. *Let Γ be a dense subgroup of a compact abelian group G. Suppose that $\{U_\gamma\}_{\gamma \in \Gamma}$ is a unitary representation of Γ in $L^2(G)$ given by a cocycle $A : \Gamma \times G \to \mathbb{T}$. Then $\{U_\gamma\}_{\gamma \in \Gamma}$ can be extended to a continuous unitary representation of G if and only if A is a coboundary.*

Proof. If A is a coboundary of the form $A(g, x) = \overline{q(x)} q(x - g)$, then define a unitary operator U_g for every g by

$$(U_g f)(x) = \overline{q(x)} q(x - g) f(x - g) \text{ for } f \in L^2(G).$$

Since the map $g \mapsto T_g f$ is continuous from G into $L^2(G)$, the mapping $g \mapsto U_g f$ is also continuous. (See the commutative diagram in Proposition 2.)

Now for the other direction of the statement we let $\{U_g\}_{g\in G}$ be a continuous unitary extension of $\{U_\gamma\}_{\gamma\in\Gamma}$. Put $A_g = U_g 1$ for every g and define $A : G \times G \to \mathbb{T}$ by $A(g,x) = A_g(x)$. It is easy to see that A is a cocycle on $G \times G$ such that $(U_g f)(x) = A(g,x)f(x-g)$ where $x \in G$, $g \in G$. Then by Proposition 3 we have an orthogonal projection P onto $\mathcal{H}_1$ in $L^2(G)$ satisfying $\int_G U_g f d\mu(g) = Pf$ for $f \in L^2(G)$. We claim that $\mathcal{H}_1 \neq \{0\}$. Suppose not. Then $\int_G U_g f d\mu(g) = 0$ for any f. Replacing f by characters χ we have

$$\int_G A(g,x)\chi(x-g)d\mu(g) = \int_G A(g,x)\chi(x)\overline{\chi(g)}d\mu(g) = 0$$

for almost every x in G. Since $|\chi(x)| = 1$ for every x, we have $\int_G A(g,x)\overline{\chi(g)}d\mu(g) = 0$ for a.e. x. Hence at a.e. fixed x, we see that $A(g,x) = 0$ in $L^2(G)$. Thus $\int_G |A(g,x)|d\mu(g) = 0$ for a.e. x and

$$\int_G \int_G |A(g,x)|dx\, d\mu(g) = \int_G \int_G |A(g,x)|d\mu(g)dx = 0.$$

Now this contradicts the fact that $A(g,x) = U_g 1(x)$ has its L^2-norm equal to 1 for every g. So our claim is proved.

Since we have $\mathcal{H}_1 \neq \{0\}$, we choose $f \in \mathcal{H}_1$ such that $||f||_2 = 1$. Then $U_g f = f$, $A(g,x)f(x-g) = f(x)$ for a.e. x. Putting $q(x) = \overline{f(x)}$, we obtain $A(g,x) = \overline{q(x)}q(x-g)$. Since $|q(x)| = |q(x)A(g,x)| = |q(x-g)|$ for every $g \in G$, we see that $|q(x)|$ is constant. Now $||q||_2 = 1$ implies that $|q(x)| \equiv 1$. □

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