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A REMARK ON p–ADIC q–BERNOULLI MEASURE

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I. Introduction

Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p–adic rational integers, the field of p–adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = p^{-1}$. We set $p^* = p$ for any prime $p > 2$ and $p^* = 4$ for $p = 2$.

Let $\bar{f} = [f, p^*]$ be the least common multiple f and p^*, and $\mathbb{Z}$ the rational integer ring.

We set

$$\mathbb{Z}_{\bar{f}} = \varprojlim \mathbb{Z}/\bar{f}p^n\mathbb{Z} \quad \text{for} \quad n \geq 0,$$

$$\mathbb{Z}_{\bar{f}}^* = \bigcup_{\substack{0<a<p\bar{f} \\ (a,p)=1}} a + \bar{f}p\mathbb{Z}_p,$$

$$a + \bar{f}p^n\mathbb{Z}_p = \{x \in \mathbb{Z}_{\bar{f}} | x \equiv a (\mod \bar{f}p^n)\},$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < \bar{f}p^n$.

When we talk q–extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p–adic number $q \in \mathbb{Q}_p$. If $q \in \mathbb{C}$, we normally assume $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume $|q-1|_p < p^{-\frac{1}{p-1}}$, so that $q^x = e^{x \log q}$ for $|x|_p < 1$. Carlitz's q–Bernoulli numbers $\beta_k = \beta_k(q)$ [1] can be determined inductively by

$$\beta_0 = 1, \quad q(q\beta + 1)^k - \beta^k = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k > 1 \end{cases}$$

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with the usual convention of replacing β^k by β_k.

Let $[x]$ be denoted by $[x] = [x : q] = \frac{1-q^x}{1-q}$.

For any positive integer N,

$$\mu_q(a + \bar{f}p^N\mathbb{Z}_p) = \frac{q^a}{[\bar{f}p^N]}.$$

is known as a distribution on $\mathbb{Z}_{\bar{f}}$ [2].

In the p-adic case [2], Carlitz's q-Bernoulli numbers $\beta_k = \beta_k(q)$ are represented by a q-analogue of Witt's formula and some properties are investigated.

In the complex case [1], the Carlitz's numbers $B^k(q) = B_k(q)$ are determined by

$$B_0(q) = 1, \ \ (qB(q)+1)^k - B_k(q) = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k > 1 \end{cases}$$

with the usual convention of replacing $B_k(q)$ by $B^k(q)$ [1],[5]. These numbers $B_k(q)$ induce Carlitz's numbers $\beta_k(q) = \beta_k$.

Note that $B_k(q) \to B_k$ as $q \to 1$. In this paper, we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{p-1}}$, so that $q^x = e^{x \log q}$ for $x \in \mathbb{Z}_p$. In [2], the p-adic q-Bernoulli polynomials $B_k(x : q)$ are represented by

$$\int_{\mathbb{Z}_p} [x+t]^n q^{-t} d\mu_q(t) = B_n(x : q)$$

Let K be a field over $\mathbb{Q}_p$. Then we call a function μ a K-measure on $\mathbb{Z}_{\bar{f}}^*$ if μ is finitely additive function defined on open-closed subsets in $\mathbb{Z}_{\bar{f}}^*$, whose values are in the field K. Any open-closed subset in $\mathbb{Z}_{\bar{f}}^*$ is disjoint union of some finite intervals $I_{a,n} = a + p^n\bar{f}\mathbb{Z}_{\bar{f}}$ in $\mathbb{Z}_{\bar{f}}^*$, where $a \in \mathbb{Z}$ prime to f, and therefore a K-measure μ is determined by its values on all the intervals in $\mathbb{Z}_{\bar{f}}$. Let $\mathbb{Q}^{(f)}$ denote the set of all rational numbers, whose denominator is a divisor of $\bar{f}p^n$ for some $n \geq 0$.

In this paper, we shall prove a q-analogue of the Nasybullin's lemma and construct a p-adic q-Bernoulli measure to obtain a p-adic q-L-function.

II. q–analogue of Nasybullin's lemma

First, we wil prove a q–analogue of Nasybullin's lemma.

THEOREM 1. *Let R be a K–valued function defined on $\mathbb{Q}^{(f)}$ with the property; there exist two constansts A,$B \in K$ such that*

$$\sum_{k=0}^{p-1} R([\frac{x+k}{p} : q^p]) = AR([x:q]) + BR([px : q^{\frac{1}{p}}]),$$
$$R([x+1:q]) = R([x:q])$$

for any number $x \in \mathbb{Q}^{(f)}$. And let $\rho \neq 0$ be a root of the equation $y^2 = Ay + Bp$. Then there exists a $K(\rho)$–measure μ on $\mathbb{Z}_{\bar{f}}^*$ such that

$$\mu(I_{a,n}) = \rho^{-n} R([\frac{a}{p^n \bar{f}} : q^{p^n \bar{f}}]) + B\rho^{-(n+1)} R([\frac{a}{p^{n-1}\bar{f}} : q^{p^{n-1}\bar{f}}])$$

for any interval $I_{a,n}$.

Note that if $q \to 1$, then the above Theorem 1 is Nasybullin' lemma [4].

Proof. It is sufficient to show that

$$\sum_{k=0}^{p-1} \mu(I_{a+p^n \bar{f}k, n+1}) = \mu(I_{a,n}).$$

Indeed,

$$\sum_{k=0}^{p-1} \mu(I_{a+p^n \bar{f}k, n+1})$$
$$= \rho^{-(n+1)} \sum_{k=0}^{p-1} R([\frac{a+p^n \bar{f}k}{p^{n+1}\bar{f}} : q^{p^{n+1}\bar{f}}])$$
$$+ B\rho^{-(n+2)} \sum_{k=0}^{p-1} R([\frac{a+p^n \bar{f}k}{p^n \bar{f}} : q^{p^n \bar{f}}])$$

$$= \rho^{-(n+1)} \sum_{k=0}^{p-1} R([\frac{k+\frac{a}{p^n \bar{f}}}{p} : (q^{p^n \bar{f}})^p])$$

$$+ B\rho^{-(n+2)} \sum_{k=0}^{p-1} R([\frac{a}{p^n \bar{f}} + k : q^{p^n \bar{f}}])$$

$$= \rho^{-(n+1)} AR([\frac{a}{p^n \bar{f}} : q^{p^n \bar{f}}]) + \rho^{-(n+1)} BR([\frac{a}{p^{n-1} \bar{f}} : q^{p^{n-1} \bar{f}}])$$

$$+ B\rho^{-(n+2)} pR([\frac{a}{p^{n-1} \bar{f}} : q^{p^n \bar{f}}])$$

$$= \rho^{-(n+2)}(\rho A + pB)R([\frac{a}{p^n \bar{f}} : q^{p^n \bar{f}}]) + \rho^{-(n+1)} BR([\frac{a}{p^{n-1} \bar{f}} : q^{p^{n-1} \bar{f}}])$$

$$= \rho^{-n} R([\frac{a}{p^n \bar{f}} : q^{p^n \bar{f}}]) + \rho^{-(n+1)} BR([\frac{a}{p^{n-1} \bar{f}} : q^{p^{n-1} \bar{f}}])$$

$$= \mu(I_{a,n}).$$

Thus

$$\mu(I_{a,n}) = \sum_{\substack{b(\mod p^{n+1}\bar{f}) \\ b \equiv a(\mod p^n \bar{f})}} \mu(I_{b,n+1}).$$

This proves our assertion because any open-closed subset is a disjoint union of some finite intervals, as already remarked above.

III. On q–Bernoulli functions

Let $B_m(x : q)$ be the mth q–Bernoulli polynomials and let $P_m([x])$ be the mth q-Bernoulli functions, that is,

$$P_m([x]) = B_m(x : q) \quad \text{for} \quad 0 \le x < 1.$$

For any real x, we easily see that $P_m([x+1]) = P_m([x])$.
Note that

$$\lim_{q \to 1} P_m([x]) = P_m(x),$$

where $P_m(x)$ is the usual Bernouli functions.

As is known [2], for any real number x

$$[p]^{m-1}\sum_{i=0}^{p-1} P_m([\frac{x+i}{p} : q^p]) = p_m([x : q]).$$

Thus the function $P_m([x])$ satisfied the property of q-Nasybullin's lemma with constants $A = [p]^{1-m}$, $B = 0$. Then $\rho \neq 0$ is equal to $[p]^{1-m}$, as $\rho^2 = A\rho + Bp$ reduces simply to $\rho^2 = [p]^{1-m}\rho$.

Therefore we obtain the following result.

THEOREM 2. *Let the function* $\mu_m = \mu_{m:q}$ *be defined on* $I_{a,n}$ *by*

$$\mu_m(I_{a,n}) = [\bar{f}p^n]^{m-1} P_m([\frac{a}{\bar{f}p^n} : q^{\bar{f}p^n}]).$$

Then μ_m *is a* $\mathbb{Q}_p(q)$*-measure on* $\mathbb{Z}^*_{\bar{f}}$.

Let χ be a primitive Dirichlet character modulo $\bar{f}$. Then the generalized q-Bernoulli number is defined in [2] by

$$\begin{aligned} B_{k,\chi}(q) &= \int_{\mathbb{Z}_{\bar{f}}} \chi(x) q^{-x}[x]^k d\mu_q(x) \\ &= [\bar{f}]^{k-1}\sum_{a=0}^{\bar{f}-1} \chi(a) B_k(\frac{a}{\bar{f}} : q^{\bar{f}}). \end{aligned}$$

We can computer a q-analogue of the p-adic L-function of Kubota-Leopoldt by the following p-adic q-Mellin-Mazur tranform [2] with respect to μ_m.

Let

$$\begin{aligned} L(\mu_m, \chi) &= \int_{\mathbb{Z}^*_{\bar{f}}} \chi(a) d\mu_m(a) \\ &= \lim_{\rho\to\infty} \sum_{\substack{a (\bmod p^\rho \bar{f}) \\ a\in\mathbb{Z}, (a,\bar{f})=1}} \chi(a)\mu_m(I_{a,\rho}). \end{aligned}$$

Since the character χ is constant on the interval $I_{a,0}$,

$$\begin{aligned} L(\mu_m, \chi) &= \sum_{a(\mod f)} \chi(a)\mu_m(I_{a,0}) \\ &= \sum_{a(\mod f)} \chi(a)[f]^{m-1} P_m([\frac{a}{f} : q^f]) \\ &= B_{m,\chi}(q) - \chi(p)[p]^{m-1} B_{m,\chi}(q^p), \end{aligned}$$

where $B_{m,\chi}(q)$ denotes the mth q-Bernoulli number containing χ.

Therefore we obtain the following result.

For $m \geq 1$,

$$\begin{aligned} \frac{-1}{m} L(\mu_m, \chi\omega^{-m}) &= \frac{-1}{m}(B_{m,\chi\omega^{-m}}(q) - \chi\omega^{-m}(p)[p]^{m-1} B_{m,\chi\omega^{-m}}(q^p)) \\ &= L_{p,q}(1-m, \chi). \end{aligned}$$

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