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ON THE REGULARITY OF THE RIEMANN MAPPING FUNCTION IN THE PLANE

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1. Introduction.

We say two regions Ω_1, Ω_2 in the plane are conformally equivalent, if there exists a conformal 1-to-1 mapping ψ of Ω_1 onto Ω_2. It follows that two conformally equivalent regions are homeomorphic. But there is a much more important relation between conformally equivalent regions: if ψ is as above, then $f \longrightarrow f \circ \psi$ is a 1-to-1 mapping of $H(\Omega_2)$ onto $H(\Omega_1)$ which preserves sums and products. Hence problems about $H(\Omega_2)$ can be transferred to problems in $H(\Omega_1)$ with the aid of the mapping ψ. In [2], Bell solved Dirichlet problem in the plane by means of the Cauchy integral. Here he used these invariant properties.

The most important case of this is based on the Riemann mapping theorem:

THEOREM R. *Given any simple connected region Ω which is not the whole plane, there exists a 1-to-1 holomorphic function f of Ω onto the unit disc Δ. Furthermore the function f is determined uniquely by conditions $f(a) = 0,\ f'(a) > 0\ a \in \Omega$.*

We call the function f in Theorem R as Riemann mapping function. In [3,4], Chung studied some of the properties for the domains in the plane using these biholomorphic mappings.

Also it is important to investigate the boundary behavior of the Riemann mapping function to study the mapping properties of the domain on the boundary. In the present work, we consider the regularity properties of the Riemann mapping functions on the boundary. That is, we will prove the following three theorems.

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THEOREM 1.1. *Suppose that Ω is a bounded simply connected region whose boundary points are simple. Then the Riemann mapping function f of Ω onto Δ extends to a 1-to-1 continuous function F of $\overline{\Omega}$ onto $\overline{\Delta}$. Futhermore, the function F maps $b\Omega$ 1-to-1 onto $b\Delta$.*

THEOREM 1.2. *Suppose that Ω is a bounded simply connected region which contains a real analytic arc $\zeta = \zeta(t)$ in its boundary. Then the Riemann mapping function $f : \Omega \longrightarrow \Delta$ extends holomorphically across ζ.*

THEOREM 1.3. *Let Ω be a simply connected region bounded by a C^∞ smooth simple closed curve. Then the Riemann mapping function f from Ω to Δ extends C^∞ smoothly upto $\overline{\Omega}$.*

Even though some of these regularity problems have been known, we present here a new and nicer method to prove the regularity theories.

2. Continuous extension.

In this section, we prove the continuous extension of the Riemann mapping function. First, we need the following lemma.

LEMMA 2.1. *Let $D \subset \mathbb{C}$ be a domain in the plane. If $f : D \longrightarrow \mathbb{C}$ is a 1-to-1 holomorphic function, then*

$$\int_D |f'|^2 \, dA = \text{Area of } f(D)$$

Proof. Let $f = u + iv$, where u and v are real valued functions on D. Then $f' = f_x = u_x + iv_x$ and by the Cauchy Riemann equation, we have

$$|f'|^2 = u_x u_x + v_x v_x = u_x v_y - u_y v_x = \det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}.$$

Hence $\int_D |f'|^2 dA = \int_{f(D)} dA =$ Area of $f(D)$.

DEFINITION 2.2. Let Ω be a simply connected region in the plane. A boundary point $b \in b\Omega$ is called simple if b has the following property: To every sequence $\{a_n\}$ in Ω converging to b, there correspond a curve $\gamma = \gamma(t)$, $0 \leq t \leq 1$, and a sequence $\{t_n\}$, $0 < t_1 < t_2 \cdots < t_n \to 1$, such that $\gamma(t_n) = a_n$ $(n = 1, 2, 3, \cdots)$ and $\gamma(t) \in \Omega$ for $0 \leq t < 1$. In other words, there exists a curve in Ω which passes through all a_n's and which ends at b.

LEMMA 2.3. *Let Ω be a bounded simply connected region. Suppose that f is a 1-to-1 holomorphic function of Ω onto Δ. Define a function ψ on $\overline{\Omega}$ by*

$$\psi(z) = \begin{cases} dist(f(z), b\Delta) & \text{if} \quad z \in \Omega \\ 0 & \text{if} \quad z \in b\Omega \end{cases}$$

Then ψ is a continuous function on $\overline{\Omega}$.

Proof. For z, w in Ω, we have

$$\begin{aligned} |\psi(z) - \psi(w)| &= \big|(1 - |f(z)|) - (1 - |f(w)|)\big| \\ &= \big||f(z)| - |f(w)|\big| < |f(z) - f(w)| \end{aligned}$$

So the continuity of ψ on Ω is clear.

To prove the continuity of ψ on $b\Omega$, suppose ψ is not continuous at $z_0 \in b\Omega$. Then there exists a sequence $\{z_n\}$ in Ω converging to z_0 such that $\psi(z_n) = dist(f(z_n), b\Delta) > \lambda > 0$ for all n, for some constant λ. Since $\{f(z_n)\}$ is bounded, there exists a subsequence $\{f(z_{n_k})\}$ of $\{f(z_n)\}$ which converges to w_0. Here w_0 must satisfy that $dist(w_0, b\Delta) \geq \lambda$, and hence that w_0 is an interior point of Ω. Let $F = f^{-1}$. Then $z_{n_k} = F(f(z_{n_k}))$ and hence

$$z_0 = \lim_{k \to \infty} z_{n_k} = \lim_{k \to \infty} F(f(z_{n_k})) = F(w_0).$$

Since F is a 1-to-1 function from Δ onto Ω and since $w_0 \in int(\Delta)$, there exists a point in Ω which is the image of w_0 under F. Thus

$$0 = \psi(z_0) = \psi(F(w_0)) = dist(w_0, b\Delta) \geq \lambda > 0$$

This is a contraction. Therefore ψ is also continuous on $b\Omega$. $\square$

Using lemma 2.3, we can prove the following theorem.

THEOREM 2.4. *Let Ω be a bounded simple connected region and let f be the Riemann mapping function of Ω onto Δ.*

(1) *If b is a simple boundary point of Ω, then f can be extended continuously to $\Omega \cup \{b\}$ and moreover $|f(b)| = 1$.*
(2) *If b_1 and b_2 are distinct simple boundary points of Ω, then f can be extended continuously to $\Omega \cup \{b_1\} \cup \{b_2\}$ and $f(b_1) \neq f(b_2)$.*

Proof. (1) Suppose that (1) is false. Then there exists a sequence $\{a_n\}$ in Ω converging to b such that $f(a_{2n}) \to w_1$, $f(a_{2n+1}) \to w_2$, $w_1 \neq w_2$

Choose a curve $\gamma = \gamma(t)$ passing through all a_n's and tending to b and set $\Gamma(t) = f(\gamma(t))$. From Lemma 2.3, $|\Gamma(t)|$ can be extended continuously to $[0,1]$ and $|\Gamma(t)| \longrightarrow 1$ as $t \longrightarrow 1$. So $|w_1| = |w_2| = 1$. Therefore it follows that one of two open arcs $\mathcal{A}$ whose union is $S - (\{w_1\} \cup \{w_2\})$ (where S is the unit circle in $\mathbb{C}$) has the property that every radius of Δ which ends at a point of $\mathcal{A}$ intersects the range of Γ in the infinite set which has a limit point on S.

Set $F = f^{-1}$. Note that F is a bounded holomorphic function on Δ and that $F(\Gamma(t)) = \gamma(t)$, $0 \leq t < 1$. Thus F has radial limits a.e, and moreover

$$\lim_{r \to 1} F(re^{i\theta}) = b \tag{2.1}$$

a.e on $\mathcal{A}$. From (2.1), we have $\lim_{r\to 1} (F - b)(re^{i\theta}) = 0$ a.e on $\mathcal{A}$, and hence $F(z) \equiv b$ on Δ because F is a holomorphic function. Note that F is 1-to-1 because f is 1-to-1. This is a contradiction and hence $w_1 = w_2$ and $|f(b)| = |w_1| = 1$.

(2) Suppose that the extension f satisfies $f(b_1) = f(b_2) = w_0$. Let γ_1 and γ_2 be curves in Ω tending to b_1 and b_2, respectively. We may assume that $dist(\gamma_1, \gamma_2) > \frac{1}{2}|b_1 - b_2|$. Let $\delta > 0$ be so small that $\Gamma_1(0) = f(\gamma_1(0))$ and $\Gamma_2(0) = f(\gamma_2(0))$ are not in $D(w_0, \delta) \cap \Delta$. Let $0 < r < \delta$. Since $\Gamma_i(t) = f(\gamma_i(t)) \to w_0$ as $t \to 1$, $i = 1, 2$, it follows that $w_1, w_2 \in D(w_0, r)$. Set $w_1 = w_0 + re^{i\theta_1}$, $w_2 = w_0 + re^{i\theta_2}$,

$-\pi \leq \theta_1, \theta_2 < \pi$, and let $\eta = \max\{|\theta_1|, |\theta_2|\}$. Then

$$\begin{aligned} |F(w_1) - F(w_2)| &= \left| \int_{\theta_1}^{\theta_2} F'(w_0 + re^{i\theta}) i r e^{i\theta} \, d\theta \right| \\ &\leq r \int_{-\eta}^{\eta} \left| F'(w_0 + re^{i\theta}) \right| \, d\theta \qquad (2.2) \\ &\leq r \left(\int_{-\eta}^{\eta} \left| F'(w_0 + re^{i\theta}) \right|^2 d\theta \right)^{\frac{1}{2}} \left(\int_{-\eta}^{\eta} d\theta \right)^{\frac{1}{2}} \end{aligned}$$

by Hölder's inequality. But by the choice of $\gamma's$, we have:

$$dist(F(w_1), F(w_2)) = |F(w_1) - F(w_2)| > \frac{1}{2}|b_1 - b_2|. \qquad (2.3)$$

From (2.2) and (2.3), we obtain an inequality

$$\frac{1}{4}|b_1 - b_2|^2 < \pi r^2 \int_{-\eta}^{\eta} \left| F'(w_0 + re^{i\theta}) \right|^2 d\theta, \qquad (2.4)$$

since $\left(\int_{-\eta}^{\eta} d\theta \right)^{\frac{1}{2}} \leq \sqrt{\pi}$. Now we integrate both sides of (2.4) with respect to r variable. Then by Lemma 2.1, it follows that

$$\begin{aligned} \infty = \int_0^{\delta} \frac{1}{4\pi r} |b_1 - b_2|^2 \, dr &\leq \int_0^{\delta} \int_{-\eta}^{\eta} \left| F'(w_0 + re^{i\theta}) \right|^2 \, r d\theta dr \\ &\leq \int_{D(w_0,\delta) \cap \Delta} |F'(s)|^2 \, dA = \text{Area of } F(D(w_0, \delta) \cap \Delta) \\ &\leq \text{Area of } \Omega < \infty, \end{aligned}$$

This is a contradiction and hence $f(b_1) \neq f(b_2)$. □

Now we are ready to prove Theorem 1.1. Here we state Theorem 1.1 again.

THEOREM 2.5. *Suppose that Ω is a bounded simply connected region whose boundary points are simple. Then the Riemann mapping* *function f of Ω onto Δ extends to a 1-to-1 continuous function of $\overline{\Omega}$ onto $\overline{\Delta}$. Futhermore, the extension f maps $b\Omega$ 1-to-1 onto $b\Delta$.*

Proof. Let z_0 be a point on $b\Omega$ and assume that $\{z_n\} \subset \overline{\Omega}$ is a sequence converging to z_0. Without loss of generality, we may assume that $\{z_n\} \subset b\Omega$. Since every boundary point of Ω is simple, it follows that for each z_n there is $z_n' \in \Omega$ so that $|z_n - z_n'| < 1/n$ and that $|f(z_n) - f(z_n')| < 1/n$. Note that $\{z_n'\}$ also converges to z_0. Therefore $f(z_n')$ converges to $f(b)$ by Theorem 2.4. Hence $f(z_n)$ also converges to $f(b)$ and this proves the continuity of f on $\overline{\Omega}$.

Note that $\Delta \subset f(\overline{\Omega}) \subset \overline{\Delta}$. Since f is continuous, it follows that $f(\overline{\Omega})$ is compact and hence $f(\overline{\Omega}) = \overline{\Delta}$. □

3. Smooth extension.

In this section, we consider the behavior of the Riemann mapping function on the region, whose boundary contains a C^∞ smooth arc or a real analytic arc. First we prove the following Generalized Schwarz Reflection Principle.

LEMMA 3.1. *Let Ω be a bounded region whose boundary contains a real analytic curve $\gamma = \gamma(t)$, $t \in (a,b)$ with $\gamma'(t) \neq 0$. Suppose that u is harmonic in Ω, continuous on $\Omega \cup \gamma$, and zero on γ. Then u extends harmonically across γ.*

Proof. Let $t_0 \in (a,b)$ and assume that γ is real analytic. So we can write $\gamma(t) = \sum_{n=0}^{\infty} \frac{\gamma^{(n)}(t_0)}{n!}(t-t_0)^n$, $t \in (t_0 - \delta, t_0 + \delta)$, for some $\delta > 0$. We replace t in the power series by a complx value z, for $|z - t_0| < \delta$ Then $\gamma = \gamma(z)$ is a holomorphic function in the disc. Consequently, we have a holomorphic function $\gamma = \gamma(z)$ in a region D, which consists of discs symmetric to the real axis and which contains the segment (a,b).

We note that $\gamma = \gamma(z)$ is locally 1-to-1 since $\gamma'(z) \neq 0$. Shink D again in which γ is 1-to-1 (again denoted by D). So we can consider $u \circ \gamma$ on D. From the properties of u and γ, it follows that $u \circ \gamma$ is defined on D and satisfies all the properties in Classical Schwarz Reflection Principle. Let D^+ be the portion that satisfies $\gamma(D^+) \subset \Omega$. Then $u \circ \gamma$ has a harmonic extension to D^- and therefore u has a harmonic extension across γ. □

LEMMA 3.2. *Let Ω be a bounded simply connected region whose boundary is a real analytic arc $\gamma = \gamma(t)$, in the sense that only one half of the disc, whose center is on γ, is inside Ω. Then every boundary point of Ω is simple.*

Proof. As in Lemma 3.1, we can regard $\gamma(t)$ as a function of complex variable t. Let $b \in b\Omega$ with $\gamma(t_b) = b$. Then there exist $D(t_b, r')$ and $D(b,r)$ $r, r' > 0$ such that each $z \in D(b,r)$ can be written as a power series $z = \sum_{n=0}^{\infty} \frac{\gamma^{(n)}(t_b)}{n!}(t - t_b)^n$ for some $t \in D(t_b, r')$ because γ is holomorphic on $D(t_b, r')$.

Let $\{a_n\}$ be a sequence in Ω such that $a_n \longrightarrow b$. Then there exists an integer $N > 0$ such that $n > N \Longrightarrow a_n \in D(b,r)$ Put $b_i = a_{N+i}$ $(i = 1, 2, \cdots)$. Then we can represent b_i as : $b_i = \sum_{n=0}^{\infty} \frac{\gamma^{(n)}(t_b)}{n!}(t_i - t_b)$ for some $t_i \in D(t_b, r')$, and it is clear that $b_i \to b$ as $i \to \infty$. Since γ is continuous on $D(t_b, r')$, it follows that $b_i \to b$ as $t_i \to t_b$.

Note that we can connect finite points by a curve. Therefore let us show that there exists a curve in Ω passing all b_i's and tending to b. That is, we assume that there is a function $\psi(s)$ $0 \leq s \leq 1$ and a sequence $\{s_i\}$ $s_1 < s_2 \cdots \to 1$ such that $\psi(s_i) = b_i$ and that $\psi(s) \in \Omega$ for $0 \leq t < 1$. In $D(t_b, r')$, two points are path connected. Therefore there exists a curve β that passes all t_i. So there is a sequence $\{s_i\}$ so that $s_1 < s_2 \cdots \to 1$ and that $\beta(s_i) = t_i$. Let $\alpha(t) = \sum_{n=0}^{\infty} \frac{\gamma^{(n)}(t_b)}{n!}(t - t_b)$. Then α is continuous on $D(t_b, r')$. Since only one half of the disc (whose center is on γ) is inside Ω, the image of the curve β under the funtion α is a curve in Ω tending to b such that $\alpha(t_i) = b_i$. If we consider $\alpha \circ \beta$, then this is the desired curve passing thru b_i's and converging to b. $\square$

Now we prove that the Riemann mapping function can be extended holomorphically to the boundary which contains a real analytic arc.

THEOREM 3.3. *Suppose that Ω is bounded by a simply connected curve which contains a real analytic arc $\zeta = \zeta(t)$. Then the Riemann mapping function $f : \Omega \longrightarrow \Delta$ extends holomorphically across ζ.*

Proof. By Lemma 3.2, f can be extended continuously to the curve ζ. Let $z_0 \in \zeta$. By Theorem 2.4, it follows that $f(z_0) \neq 0$. We choose a disc D centered at z_0 and that $f(z) \neq 0$ on $\overline{D}$. Let us consider the function $h(z) = \log|f(z)|$ on D. Then $h(z)$ is harmonic on D^+ where D^+ is the subset of D in Ω. Since f is continuous on $\overline{D^+}$,

$h(z) = \log|f(z)|$ is also continuous on $\overline{D^+}$. Note that $\overline{D^+} = D^+ \cup \zeta_D$, where ζ_D is the part of ζ in D. Moreover $h \equiv 0$ on ζ_D, since $|f(z)| = 1$ on ζ. Hence by the Generalized Schwarz Reflection Principle, h extends harmonically accross ζ. Shink disc again so that h is harmonic on $D(z_0, \varepsilon)$. Let v be the harmonic conjugate of h on $D(z_0, \varepsilon)$ and set $H(z) = h(z) + iv(z)$. Then H is holomorphic on $D(z_0, \varepsilon)$ and satisfies

$$|e^{H(z)}| = e^{h(z)} = |f(z)|.$$

Hence

$$\frac{f(z)}{e^{H(z)}} = e^{i\theta}, \quad \text{or} \quad f(z) = e^{i\theta}e^{H(z)},$$

for some $\theta \in [0, 2\pi)$. We note that $e^{i\theta}e^{H(z)}$ is holomorphic and hence that $e^{i\theta}e^{H(z)}$ extends f holomorphically in $D(z_0, \varepsilon) \cap \overline{\Omega}$. □

THEOREM 3.4. *Let Ω be a simply connected region bounded by a C^∞ smooth simple closed curve. Then the Riemann mapping function f from Ω to Δ extends C^∞ smoothly to $\overline{\Omega}$.*

Proof. Note that all of the boundary points of the region that is bounded by a C^∞ smooth simple closed curve are simple. Hence f extends continuously upto the boundary by Theorem 2.5. Let $a \in \Omega$ and $f(a) = 0$. Define a function $\psi : \Omega \longrightarrow \mathbb{R}$ as follows: $\psi(z) = \log|f(z)|$ for all z except a and $\psi(a) = 0$. Since $\psi(z) = \log|f(z)|$ is the real part of $\log f(z)$ which is analytic in $\Omega - \{a\}$, ψ is harmonic in $\Omega - \{a\}$. From Lemma 2.3, it follows that $dist(f(z), b\Delta) = 1 - |f(z)|$ is continuous on $\overline{\Omega}$, and equal to zero on $b\Omega$. Thus $\log|f(z)|$ is also continuous on $\overline{\Omega} - \{a\}$ and equal to zero on $b\Omega$.

Choose a small $\varepsilon > 0$ so that $\overline{D}(a, \varepsilon) \subset \Omega$. Choose a C^∞ cut-off function ξ defined on $\overline{\Omega}$ so that $\xi \equiv 0$ on $\overline{D}(a, \frac{\varepsilon}{2})$ and $\xi \equiv 1$ in $D^c(a, \varepsilon)$. Set $u(z) = \xi(z)\psi(z) = \xi(z)\log|f(z)|$, $z \in \overline{\Omega}$. Then it is clear that u is continuous on $\overline{\Omega}$, equal to zero on $b\Omega$ and that $\Delta u \in C^\infty(\overline{\Omega})$. This last property shows that u itself is smooth on $\overline{\Omega}$ by the regularity theory of the Laplace operator [2]. Note that $u(z) = \log|f(z)|$ near $b\Omega$. Thus $\log|f(z)|$ extends C^∞ smoothly upto $b\Omega$ and so does

$$\left(e^{\log|f(z)|}\right)^2 = |f(z)|^2 = f(z)\bar{f}(z).$$

Thus f is a smooth function on $\overline{\Omega}$ and this proves our theorem. □

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