Nonequilibrium Phenomena in Globally Coupled Active Rotators with Multiplicative and Additive Noises

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CONTENTS

ABSTRACT

We investigate noise-induced phase transitions in globally coupled active rotators with multiplicative and additive noises. In the system there are four phases, stationary one-cluster, stationary two-cluster, moving one-cluster, and moving two-cluster phases. It is shown that multiplicative noise induces a bifurcation from one-cluster phase to two-cluster phase. Pinning force also induces a bifurcation from moving phase to stationary phase suppressing the multiplicative noise effect. Additive noise reduces both effects of multiplicative noise and pinning force urging the system to the stationary one-cluster phase. The frustrated effects of pinning force and additive and multiplicative noises lead to a reentrant transition at intermediate additive noise intensity. Nature of the transition is also discussed.

I. INTRODUCTION

In sensory processing the linking of sensory inputs across multiple receptive fields is a fundamental task to identify distinct objects, segment them from each other, and separate them from background. This linkage is difficult to approach within the framework of most current artificial neural network models because they use only the levels of activity in individual neurons to encode information. To overcome this difficulty the model has been suggested in which global properties of stimuli are identified through correlations in the temporal firing patterns of different neurons [1]. Recent experiments provided support to this concept showing that neurons in the primary visual cortex of the cat can exhibit oscillatory responses [2], [3]. The responses are coherent over relatively large distances and are sensitive to global properties of stimuli. The existence of temporal synchronization over relatively large distances in the cortex suggests that the processing of information is a cooperative process of neurons with different receptive fields.

Recently the complex oscillating neural network model was suggested as a model for flexible pattern recognition [4], [5]. The model is based on the self-organization of a spatio-temporal pattern in an oscillating network. The flexibility of pattern recognition is caused from the flexibility of connections by entrainment and from the stability of dynamical patterns for deformation. Although the model shows the possibility of flexible pattern recognition by the oscillating neural network, the dynamics of the system is unclear because of its complexity. A simple model is necessary to investigate dynamics of oscillating neural networks.

Oscillations of neuronal activity in the visual cortex and their potential role in computation have been the topic of much recent investigation. A coupled phase oscillator model which consists of neurons with oscillatory outputs was suggested to understand the temporal and spatial coherence of the oscillations [6]. The coupled phase oscillators have been studied extensively as a model system to understand dynamics of various systems such as Josephson-junction arrays [7], chemical reactions [8], charge-density waves [9], [10] and phased antenna arrays [11]. In the weak coupling limit dynamics of the coupled phase oscillators has usually been investigated in the reduced model with the effective interaction given by the first Fourier mode [12]-[15]. It has been claimed, however, that higher Fourier mode interactions are indispensable for interesting collective dynamics [16], [17]. It has been also shown that the phase oscillators with the interaction of higher harmonics eventually converged to the clustered states at some parameter range.

Coupled active rotators have been studied as a phase model of either coupled limit-cycle oscillators or coupled excitable elements [18]. Particularly the question on the role of noise in coupled rotator models has been raised continuously. Transitions induced by multiplicative noise in low dimensional dynamical systems are by now a familiar phenomena [19], but multiplicative noise in spatially distributed and/or high dimensional systems remains the focus of current research [20]. The globally coupled active rotators with additive noise show the transition from moving (excited) state to stationary (inhibited) state at a critical noise intensity. Recently, we showed that in the globally coupled active rotator model with randomly fluctuating interaction, multiplicative noise induced an interesting nonequilibrium phenomenon [21]-[23]: at a critical noise intensity the system undergoes a noise-induced phase transition and is split into clusters both in stationary and moving phases. This is a pure multiplicative noise effect and shows a route to the clustering phenomena without introducing higher Fourier mode interactions which have usually been considered to be necessary for clustering. It was also shown that there exists a reentrant transition in the presence of multiplicative and additive noises. The reentrant transition is an interplay effect of multiplicative and additive noises.

In this paper we study extensively the nonequilibrium phenomena of the globally coupled active rotators induced by interplay between multiplicative and additive noises. In the computational point of view, the additive noise plays a role of regulator to avoid the singularity of probability distribution of the Fokker-Planck equation [24]. The multiplicative noise, in the presence of additive noise, induces a bifurcation at a critical value of multiplicative noise intensity, thus forming a stable two-cluster state. On the other hand, the additive noise and the external source term suppress the effect of the multiplicative noise on the system leading to frustration. Balancing this frustration, the system reveals various interesting phase portraits such as a reentrant transition at an intermediate additive noise intensity. Nature of the transition is also discussed.

In the following section we describe the model under study in this paper. Section III and IV are devoted to present the analytical and numerical studies, respectively. Implication of the nonequilibrium phenomena of the system is discussed with summarized results in Sec. V.

II. MODEL

A (noiseless) model of N coupled active rotators under study is expressed by the equation of motion [18], [21]-[23], [25]

$$\frac{d\phi_i}{dt} = \omega - b\sin\phi_i - \sum_{j=1}^{N} K_{ij}\sin(\phi_i - \phi_j)\,, \qquad (1)$$

where ϕ_i, $i = 1,\ 2,\ \cdots,\ N$, is the phase of the ith rotator. ω is the intrinsic frequency that is uniformly given to each rotator. The second term on the right-hand side of (1) (from now on we denote this as the b term) is introduced to mimic the dynamics of stochastic limit-cycle oscillators or excitable elements [18], [21]-[23], [25]. The third term on the right-hand side of (1) describes global coupling, which depends on the phase difference of two rotators. If the coupling is excitatory, i.e. $K_{ij} > 0$, then this term gives perfect synchrony, which means $\phi_i(t) = \phi(t)$ for all i. In the steady state ϕ_i dwells on two phases. When $|b/\omega| > 1$, each element is at the stable fixed point, $\phi_i = \phi_0 \equiv \sin^{-1}(\omega/b)$. When $|b/\omega| < 1$, the system is on the moving phase, i.e., each ϕ_i is a rotator with frequency $\sqrt{\omega^2 - b^2}$. The b term of (1) characterizes the system whether it is on the stationary state or on the moving state.

Now we assume the uniform excitatory interaction $K_{ij} = K/N > 0$. If the system is coupled to a fluctuating environment, the coupling strength then may be assumed to be a stochastic quantity, which implies

$$\frac{K}{N} \longrightarrow \frac{1}{N}(K + \sigma_M \eta_i(t)), \tag{2}$$

where $\eta_i(t)$ is a Gaussian white noise characterized by

$$\begin{aligned} \langle \eta_i(t) \rangle &= 0, \\ \langle \eta_i(t)\eta_j(t') \rangle &= \delta_{ij}\delta(t - t'), \end{aligned} \tag{3}$$

and σ_M measures the intensity of the multiplicative noise. The Gaussian white noise with mean zero in the coupling yields the inhibitory interactions as well as the excitatory ones. The system therefore has an approximate symmetry $\phi_i \to \phi_i + \pi$ when σ_M is large. When the symmetry is exact, i.e., in the $\sigma_M \to \infty$ limit, one expects an equal intensity of the two clusters located at $\phi = 0$ and π. In the presence of the multiplicative noise the system shows bifurcation from one-cluster state to two-cluster state both in stationary and moving phases. This noise-induced transition provides a route to clustering phenomena without introducing higher Fourier mode interactions, which can not be seen in the deterministic case or in the system with a simple additive noise.

For small multiplicative noise intensity the perfect synchrony of the system persists leading to the singularity of the probability distribution of the Fokker-Planck equation corresponding to the equation of motion of the system. To remove the singularity we introduce an additive noise $\xi_i(t)$ to the system. The interplay between additive and multiplicative noises also induces interesting nonequilibrium phenomena such as a reentrant transition. Thus in the presence of additive noise as well as multiplicative noise, (1) is replaced by the stochastic differential equation

$$\begin{aligned} \frac{d\phi_i}{dt} = \omega - b\sin\phi_i - \frac{1}{N}(K + \sigma_M \eta_i(t)) \\ \times \sum_{j=1}^{N} \sin(\phi_i - \phi_j) + \sigma_A \xi_i(t), \end{aligned} \tag{4}$$

where $\xi_i(t)$ is a Gaussian white noise independent to $\eta_i(t)$'s. $\xi_i(t)$ is characterized by

$$\begin{aligned} \langle \xi_i(t) \rangle &= 0, \\ \langle \xi_i(t)\xi_j(t') \rangle &= \delta_{ij}\delta(t - t'), \\ \langle \xi_i(t)\eta_j(t') \rangle &= 0, \end{aligned} \tag{5}$$

and σ_A measures the intensity of the additive noise. Throughout this paper we set $K = 1$ using a suitable time unit.

III. ANALYTICAL STUDY

The macroscopic behavior of the system can be described by the probability distribution

$P(\phi,\ t)$ of ϕ_i at time t, whose evolution is governed by the Fokker-Planck equation [26]. In the large N limit (4) yields the Fokker-Planck equation

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial \phi}\left[\left\{\omega - b\sin\phi - \int_0^{2\pi} d\phi' \sin(\phi-\phi') \times n(\phi',t) + \frac{\nu\sigma_M^2}{2}\int_0^{2\pi} d\phi' \sin(\phi-\phi') n(\phi',t) \times \int_0^{2\pi} d\phi'' \cos(\phi-\phi'') n(\phi'',t)\right\} P(\phi,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial\phi^2}\left[\left\{\sigma_A^2 + \sigma_M^2\left(\int_0^{2\pi} d\phi' \sin(\phi-\phi') \times n(\phi',t)\right)^2\right\} P(\phi,t)\right], \tag{6}$$

with $\nu = 1$ for Stratonovich interpretation and $\nu = 0$ for Itô interpretation. In (6) $n(\phi,t)$, the normalized number density of the rotators with phase ϕ at time t, is given by

$$n(\phi,\ t) = \frac{1}{N}\sum_{i=1}^{N} \delta(\phi_i(t) - \phi)\,. \tag{7}$$

Since ϕ_i's are statistically independent for the uniform interaction, $P(\phi,t)$ may be identified with $n(\phi,t)$. In this paper we'll analyze the steady state of $n(\phi,t)$.

When $\omega = 0$ the steady state of the system is a stationary state, i.e., $\partial P/\partial t = 0$. In this case the Fokker-Planck equation (6) can be solved self-consistently [23] leading to

$$n(\phi) = Z^{-1}(1 + A\cos\phi)^{\gamma-(2-\nu)/2} \times (1 - A\cos\phi)^{-\gamma-(2-\nu)/2}, \tag{8}$$

where

$$\gamma = \frac{b+\Delta}{\sigma_M\Delta\sqrt{\sigma_A^2 + \sigma_M^2\Delta^2}}, \qquad A = \frac{\sigma_M\Delta}{\sqrt{\sigma_A^2 + \sigma_M^2\Delta^2}} \tag{9}$$

with self-consistent equation

$$\Delta = \int_0^{2\pi} \cos\phi\, n(\phi) d\phi\,. \tag{10}$$

In (8) Z is given by the normalization condition, $\int_0^{2\pi} n(\phi) d\phi = 1$. A detailed derivation of the calculation has been presented elsewhere [23].

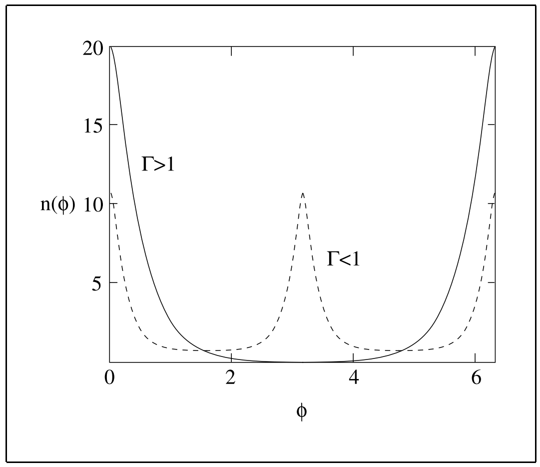

Fig. 1. Plot of $n(\phi)$ as a function of ϕ: solid line for $\Gamma > 1$ ($\sigma_M = 4.2\sigma_A$ and $b = 4.9\sigma_A^2 - 0.72$) and dotted line for $\Gamma < 1$ ($\sigma_M = 610\sigma_A$ and $b = 49\sigma_A^2 - 0.016$).

$n(\phi)$ given by (8) has a maximum at $\phi = 0$ or 2π. With the definition of $\Gamma \equiv 2\gamma/(2-\nu)A$, (8) shows that for $\Gamma > 1$ $n(\phi)$ has a minimum at $\phi = \pi$, and for $\Gamma < 1$ $n(\phi)$ has a local maximum at $\phi = \pi$ and a minimum at $\phi = \cos^{-1}(-\Gamma)$ (Fig. 1). We identify multiple peaks in the distribution as corresponding to multiple clusters of like-phased rotators, and interpret the distribution as the instantaneous distribution of rotator phases, rather than as the distribution over time of the average phase. These are two physically distinct interpretations, and which one is correct can be (and was) checked in actual direct simulations of the globally coupled system. We found that the cluster interpretation is valid (Fig. 3). Thus the critical point is given by $\Gamma = 1$ implying a continuous transition from a one-cluster state to a two-cluster state at that point: While for $\Gamma < 1$ $n(\phi)$ has a peak representing a one-cluster state, for $\Gamma > 1$ it has two peaks representing a two-cluster state.

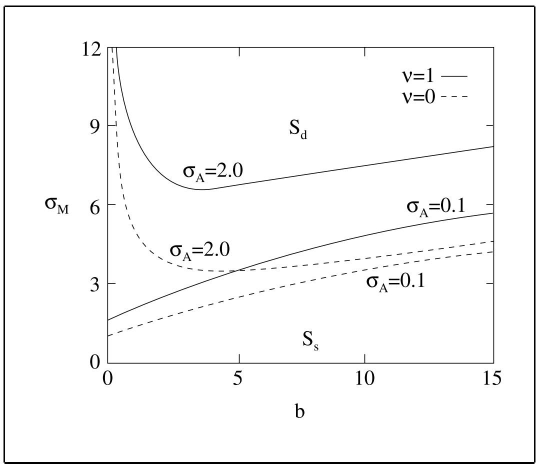

Fig. 2. Phase diagrams for Stratonovich (solid lines) and Itô (dashed lines) interpretations in σ_M-b plane at various values of σ_A. S_s and S_d represent one-cluster and two-cluster phases, respectively.

Figure 2 shows the phase diagrams for various values of σ_A when $\omega = 0$. The figure also shows that for small σ_A critical value of σ_M increases monotonically as b increases. This behavior implies that b term suppresses the multiplicative noise effect on the system which tends to split the rotators into two clusters. The behavior can be understood easily because b term gives pinning effect at $\phi = 0$ and depinning effect at $\phi = \pi$. For large σ_A there is a reentrant transition as b decreases. At large b the system is in a one-cluster stationary (S_s) state, and as b decreases the system goes to a two-cluster stationary (S_d) state, and as b decreases further the system reenters the S_s state. This reentrant transition comes from the interplay of additive and multiplicative noises. The additive noise suppresses effects of the b term as well as the multiplicative noise. Thus the effects of additive and multiplicative noises and b term are frustrated. These frustrated effects result in the reentrant transition.

IV. NUMERICAL STUDY

For finite ω equation of motion of the system considered here can be written as

$$\frac{d\phi_i}{dt} = \omega - b\sin\phi_i - C(\{\phi_j\})\sin\phi_i + S(\{\phi_j\})\cos\phi_i \\ -(C(\{\phi_j\})\sin\phi_i - S(\{\phi_j\})\cos\phi_i)\sigma_M\eta_i(t) \\ +\sigma_A\xi_i(t)\,, \tag{11}$$

where

$$C(\{\phi_j\}) = \frac{1}{N}\sum_{i=1}^{N}\cos\phi_i$$
$$S(\{\phi_j\}) = \frac{1}{N}\sum_{i=1}^{N}\sin\phi_i\,. \tag{12}$$

Here $\{\phi_j\}$ represents $\{\phi_1, \phi_2, \cdots, \phi_N\}$. Since $\eta_i(t)$ and $\xi_i(t)$ are independent Gaussian white noises we can replace the noises by a Gaussian white noise $\zeta_i(t)$ as

$$-(C(\{\phi_j\})\sin\phi_i - S(\{\phi_j\})\cos\phi_i)\sigma_M\eta_i(t) + \sigma_A\xi_i(t) \\ \to \sqrt{(C(\{\phi_j\})\sin\phi_i - S(\{\phi_j\})\cos\phi_i)^2\sigma_M^2 + \sigma_A^2}\,\zeta_i(t)\,. \tag{13}$$

Here $\zeta_i(t)$ is characterized by

$$\langle \zeta_i(t) \rangle = 0\,,$$
$$\langle \zeta_i(t)\zeta_j(t')\rangle = \delta_{ij}\delta(t-t')\,. \qquad (14)$$

Then the equation of motion (11) is replaced by

$$\frac{d\phi_i}{dt} = h_i(\{\phi_j\}) + g_i(\{\phi_j\})\zeta_i(t) \qquad (15)$$

with

$$h_i(\{\phi_j\}) = \omega - b\sin\phi_i - C(\{\phi_j\})\sin\phi_i + S(\{\phi_j\})\cos\phi_i\,,$$
$$g_i(\{\phi_j\}) = \sqrt{(C(\{\phi_j\})\sin\phi_i - S(\{\phi_j\})\cos\phi_i)^2\sigma_M^2 + \sigma_A^2}\,. \qquad (16)$$

To investigate phase transitions at finite ω, we have performed numerical simulation of (15). In the simulation, we have used the efficient Runge-Kutta method based on the Stratonovich interpretation [27] with discrete time steps of $\Delta t = 0.01$ and random initial configurations. The efficient Runge-Kutta method used here is given by

$$\begin{aligned}
h_{i0} &= h_i(\{\phi_j(t_n)\})\,,\\
g_{i0} &= g_i(\{\phi_j(t_n)\})\,,\\
g_{i1} &= g_i(\{\phi_j(t_n) + \tfrac{1}{2}g_{j0}\zeta_j(t_n)\sqrt{\Delta t}\})\,,\\
g_{i2} &= g_i(\{\phi_j(t_n) + \tfrac{1}{4}h_{j0}(3\Delta t + \zeta_j^2(t_n)\Delta t) + \tfrac{1}{2}g_{j1}\zeta_j(t_n)\sqrt{\Delta t}\})\,, \qquad (17)\\
g_{i3} &= g_i(\{\phi_j(t_n) + \tfrac{1}{2}h_{j0}(3\Delta t - \zeta_j^2(t_n)\Delta t) + g_{j2}\zeta_j(t_n)\sqrt{\Delta t}\})\,,\\
h_{i1} &= h_i(\{\phi_j(t_n) + \tfrac{1}{2}h_{j0}(3\Delta t - \zeta_j^2(t_n)\Delta t) + g_{j2}\zeta_j(t_n)\sqrt{\Delta t}\})\,,\\
\phi_i(t_{n+1}) &= \phi_i(t_n) + \tfrac{1}{2}(h_{i0}+h_{i1})\Delta t + \tfrac{1}{6}(g_{i0} + 2g_{i1} + 2g_{i2} + g_{i3})\zeta_i(t_n)\sqrt{\Delta t}\,,
\end{aligned}$$

with discrete time $t_n \equiv n\Delta t$ of integer n where $\zeta_j(t_n)$'s are independent Gaussian random numbers with mean zero and variance one, characterized by

$$\langle \zeta_i(t_n)\rangle = 0,$$
$$\langle \zeta_i(t_n)\zeta_j(t_{n'})\rangle = \delta_{ij}\delta_{nn'}\,. \qquad (18)$$

At each run, the first 4×10^4 time steps per rotator have been discarded to achieve steady state and 10^5 time steps per rotator have been used to compute averages. We have considered the system of size $N = 1000$.

For finite ω the system has four phases, stationary one-cluster (S_s), stationary two-cluster (S_d), moving one-cluster (P_s), and moving two-cluster (P_d) phases. Figure 3 shows the time evolutions of $n(\phi, t)$ at steady state in the four phases. In S_d and P_d phases there exist two stable clusters of rotators. The locations of two clusters differ by π. The Gaussian white noise with mean zero in the coupling yields the inhibitory interaction as well as the excitatory one. The system therefore has an approximate symmetry $\phi_i \to \phi_i + \pi$ when σ_M is large. When this symmetry is exact, i.e. in the $\sigma_M \to \infty$ limit, one expects the equal intensity of the two clusters.

To characterize phases of the system we have calculated the temporal fluctuation of $C(t)$, ΔC, and the steady-state correlation function, $n_c(\phi)$, defined by

$$\Delta C = \langle C^2(t)\rangle_t - \langle C(t)\rangle_t^2\,,$$
$$n_c(\phi) = \int_0^{2\pi} d\phi' \langle n(\phi', t)n(\phi+\phi', t)\rangle_t\,, \qquad (19)$$

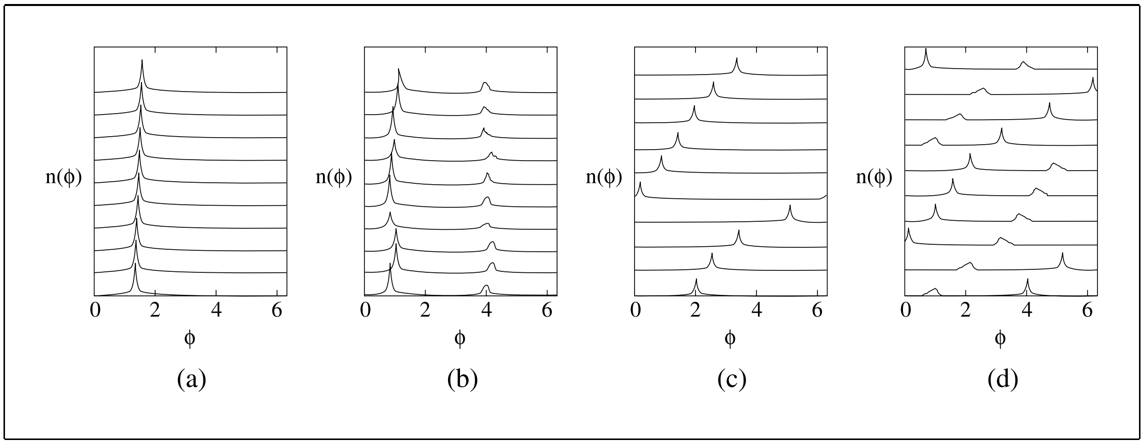

Fig. 3. Time evolutions of $n(\phi,t)$ at various phases in steady state with unit time interval: (a) at $\omega=1$, $b=0.9$, $\sigma_M=3$, and $\sigma_A=0.1$ in stationary one-cluster phase (S_s), (b) at $\omega=1$, $b=0.5$, $\sigma_M=20$, and $\sigma_A=0.1$ in stationary two-cluster phase (S_d), (c) at $\omega=1$, $b=0.5$, $\sigma_M=3$, and $\sigma_A=0.1$ in moving one-cluster phase (P_s), (d) at $\omega=1$, $b=0.1$, $\sigma_M=10$, and $\sigma_A=0.1$ in moving two-cluster phase (P_d).

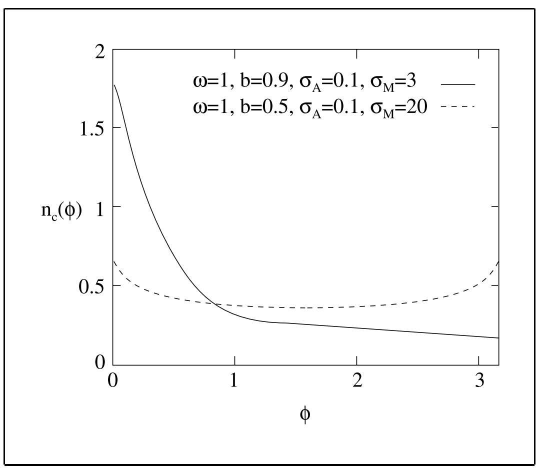

Fig. 4. Steady-state correlation function $n_c(\phi)$ of rotator number density in S_s (solid line for $\omega=1$, $b=0.9$, $\sigma_M=3$, and $\sigma_A=0.1$) and S_d (dashed line for $\omega=1$, $b=0.5$, $\sigma_M=20$, and $\sigma_A=0.1$) phases. By definition $n_c(2\pi-\phi)=n_c(\phi)$.

respectively, where $\langle\cdots\rangle_t$ represents a time average. Figure 4 shows the steady-state correlation function at S_s and S_d phases. While the correlation function $n_c(\phi)$ has a peak at $\phi=0$ in S_s phase, it has two peaks at $\phi=0$ and π in S_d phase. In the P_s and P_d phases, the correlation function has also a peak and two peaks, respectively. The phases of the system are characterized by the criterions shown in Table 1.

Figure 5 shows phase diagrams in σ_A-b plane for various values of ω with $\sigma_M=3$. In Fig. 5(a) we show the phase diagram for $\omega=0$. When $\sigma_A=0$, the system has a transition point at a critical value of b, $b_c\sim3.5$. For $b<b_c$, the system is on S_d phase due to multiplicative noise. For $b>b_c$, it is on S_s phase implying b term suppresses the multiplicative noise effect on the system. As σ_A increases, the phase structure persists up to some critical value of σ_A, $\sigma_{c1}\sim0.84$, reducing the value of the transition point b_c. For $\sigma_{c1}<\sigma_A<\sigma_{c2}\sim0.94$,

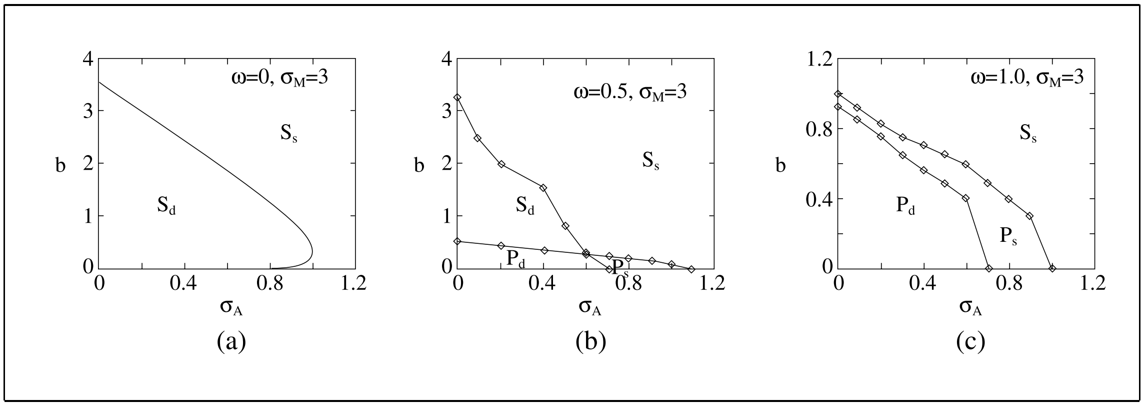

Fig. 5. Phase diagrams in σ_A-b plane with $\sigma_M = 3$. (a) has been obtained from the analytical solution Eq. (8) for $\omega = 0$. (b) and (c) have been obtained by numerical simulation performed for the system of size $N = 1000$ for $\omega = 0.5$ and 1, respectively. Lines in (b) and (c) are merely guides to eyes. S_s (P_s) and S_d (P_d) represent stationary (moving) one-cluster and two-cluster phases, respectively.

Table 1. Criterions to characterize phases of the globally coupled active rotators. n_p represents the number of peaks of steady-state correlation function $n_c(\phi)$.

phase	ΔC	n_p
S_s	zero	one
S_d	zero	two
P_s	nonzero	one
P_d	nonzero	two

the system shows a reentrant transition, i.e., at small b the system reenters into S_s phase. This reentrant transition results from the frustration effect between b term and additive noise because additive noise suppresses both effects of b term and multiplicative noise. For $\sigma_A > \sigma_{c2}$, the system is on S_s phase regardless of the value of b implying additive noise suppresses entirely the multiplicative noise effect.

Figure 5(b) shows the phase diagram for $\omega = 0.5$. When $\sigma_A = 0$, there are two transition points at critical values of b, $b_{c1} \sim 0.49$ and $b_{c2} \sim 3.25$. For $b < b_{c1}$, ω drives the system to moving state leading to P_d phase. For $b_{c1} < b < b_{c2}$, b term pins the system to a fixed point giving S_d phase. For $b > b_{c2}$, b term is large enough to induce S_s phase suppressing entirely the multiplicative noise effect. As σ_A increases, the phase structure persists up to some critical value of σ_A, $\sigma_{c1} \sim 0.6$, reducing the values of the transition points b_{c1} and b_{c2}. At $\sigma_A = \sigma_{c1}$, the S_d phase shrinks and as σ_A increases further P_s phase appears giving two transition points b_{c3} and b_{c4} at which the phase transitions from P_d to P_s and from P_s

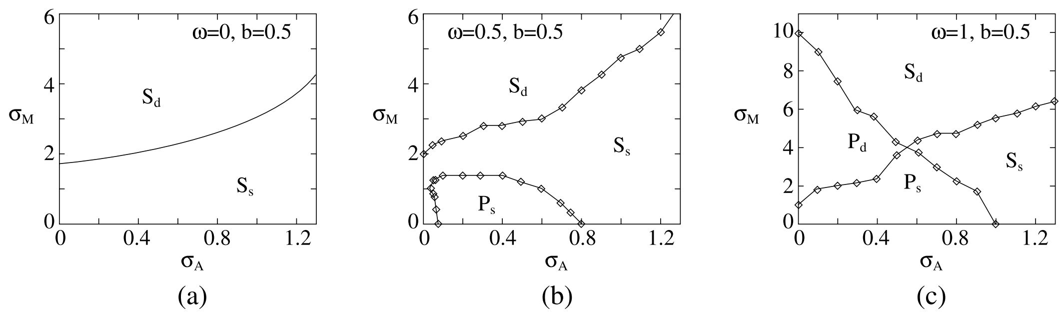

Fig. 6. Phase diagrams in σ_A-σ_M plane with $b=0.5$. (a) has been obtained from the analytical solution Eq. (8) for $\omega=0$. (b) and (c) have been obtained by numerical simulation performed for the system of size $N=1000$ for $\omega=0.5$ and 1, respectively. Lines in (b) and (c) are merely guides to eyes. S_s (P_s) and S_d (P_d) represent stationary (moving) one-cluster and two-cluster phases, respectively.

to S_s occur, respectively. The transition points b_{c3} and b_{c4} also decrease as σ_A increases. At $\sigma_A=\sigma_{c2}\sim 0.7$ the P_d phase shrinks leading to a single transition from P_s to S_s at b_{c4}. For $\sigma_A>\sigma_{c2}$, the additive noise is large enough to take off the two-cluster phase suppressing the multiplicative noise effect. As σ_A increases further, b_{c4} decreases vanishing at $\sigma_A=\sigma_{c3}\sim 1.1$. For $\sigma_A>\sigma_{c3}$ the system is on the S_s phase for all b implying that the additive noise pins the system to a single cluster state. In contrast to the case of $\omega=0$, when $\omega=0.5$, there is no reentrant transition. This result comes from that the driving force ω relaxes the frustration effect between additive noise and b term.

Figure 5(c) shows the phase diagram for $\omega=1$. When $\sigma_A=0$, there are two transition points at critical values of b, $b_{c1}\sim 0.95$ and $b_{c2}\sim 0.99$. For $b<b_{c1}$, the system is on the P_d phase due to driving force ω and multiplicative noise σ_M. For $b_{c1}<b<b_{c2}$, b term dominates the multiplicative noise effect leading to the P_s phase. For $b>b_{c2}$, b term pins the system to a fixed point giving S_s phase. In the contrary to the case of $\omega=0.5$, when $\omega=1$, b term dominates the multiplicative noise effect before it pins the system to a fixed point. This is because b must dominate the driving force ω to pin the system at a fixed point. As σ_A increases, the phase structure persists up to some critical value of σ_A, $\sigma_{c1}\sim 0.7$, at which b_{c1} shrinks to zero. For $\sigma_{c1}<\sigma_A<\sigma_{c2}\sim 1.0$, the system has a transition point from P_s phase to S_s phase. For $\sigma_A>\sigma_{c2}$, the system is on the S_s phase for all b. In contrast to the case of simple additive noise case, which has been studied by Shinomoto and Kuramoto [18], there is no nonanalyticity in the phase boundary. Rather, it continues to the infinite value of σ_M.

Figure 6 shows phase diagrams in σ_A-σ_M

plane for various values of ω with $b = 0.5$. In Fig. 6(a) we show the phase diagram for $\omega = 0$. When $\sigma_A = 0$, the system has a transition point at a critical value of σ_M, $\sigma_{Mc} \sim 1.7$. For $\sigma_M < \sigma_{Mc}$, the system is on S_s phase by pinning due to b term. The multiplicative noise induces a bifurcation from stationary one-cluster phase to stationary two-cluster phase at $\sigma_M = \sigma_{Mc}$ above which the system is on S_d phase. As σ_A increases, the phase structure persists increasing the value of σ_{Mc}. This implies that additive noise suppresses the multiplicative noise effect.

Figure 6(b) shows the phase diagram for $\omega = 0.5$. When $\sigma_A = 0$, the system has a transition point from S_s phase to S_d phase at a critical value of σ_M, $\sigma_{Mc1} \sim 2.05$. As σ_A increases, the phase structure persists up to some critical value of σ_A, $\sigma_{c1} \sim 0.03$, increasing σ_{Mc1}. For $\sigma_{c1} < \sigma_A < \sigma_{c2} \sim 0.07$, P_s phase appears at intermediate σ_M showing a reentrant transition from P_s into S_s at small σ_M. This P_s phase results from the suppression of b and σ_A effects on the system due to σ_M. The reentrant region shrinks as σ_A increases finally vanishing at $\sigma_A = \sigma_{c2}$. For $\sigma_{c2} < \sigma_A < \sigma_{c3} \sim 0.8$, the system has two transition points from P_s to S_s and from S_s to S_d at critical values of σ_M, σ_{Mc1} and σ_{Mc2}, respectively. As σ_A increases above σ_{c3}, only one transition point, σ_{Mc1}, from S_s to S_d exists.

Figure 6(c) shows the phase diagram for $\omega = 1$. When $\sigma_A = 0$, there are two transition points at critical values of σ_M, $\sigma_{Mc1} \sim 1.05$ and $\sigma_{Mc2} \sim 10.0$. For $\sigma_M < \sigma_{Mc1}$, the system is on the P_s phase due to driving force ω. At $\sigma_M = \sigma_{Mc1}$, there is a bifurcation from moving one-cluster phase to moving two-cluster phase due to the multiplicative noise. For $\sigma_M > \sigma_{Mc2}$, the system is on the stationary two-cluster phase because multiplicative noise is large enough to pin the system to a fixed point. As σ_A increases, the phase structure persists up to some critical value of σ_A, $\sigma_{c1} \sim 0.55$, increasing σ_{Mc1} and decreasing σ_{Mc2}. At $\sigma_A = \sigma_{c1}$, σ_{Mc1} meets σ_{Mc2} shrinking the P_d phase. As σ_A increases above σ_{c1}, S_s phase appears implying the multiplicative noise intensity to split the system into two clusters is larger than that to pin the system to a fixed point. As σ_A increases further, S_s phase swells shrinking P_s phase. Finally, at $\sigma_A \sim 1.0$ P_s phase disappears leading to a single transition point.

V. CONCLUSION

In this paper we considered the nonequilibrium phenomena in globally coupled active rotators with additive and multiplicative noises. We showed that the multiplicative noise induced the bifurcation from one-cluster state to two-cluster state at a critical intensity of the multiplicative noise. While driving force ω drives the system to move, b term plays a role of pinning force which pins the system to a fixed point. The cooperation of multiplicative noise, ω, and b term leads to four phases of the system, stationary (moving) one-cluster and two-cluster phases. Since b term gives pinning force at $\phi = 0$ and depinning force at $\phi = \pi$, it suppresses the multiplicative noise effect on the system. Additive noise suppresses all effects of the multiplicative noise, ω, and b term leading to frustration. This frustrated effect provides various interesting nonequilibrium phenomena such as a reentrant transition. We showed the phase diagrams of the system for various parameter values.

The active rotator model was presented as a phenomenological model of oscillatory neural network. The dynamics of each neuron is described by a phase variable. This description does not consider the origin of the oscillations and oversimplifies the neuronal dynamics. Nevertheless, such an approach is justified at present, because of the lack of experimental guidance on the neuronal circuitry. We introduced the multiplicative noise into the system to simulate the fluctuating synaptic coupling. The splitting of the system into two clusters induced by the multiplicative noise implies the segmentation of visual scene in visual cortex. Therefore, our results exhibit another route to the clustering and the segmentation phenomena, which cannot be seen in the deterministic case or in the system with only a simple additive noise. It would be interesting if our results can be tested in physiological systems.

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REFERENCES

[1] C. von der Malsburg and W. Schneider, "A neural cocktail-party processor," *Biol. Cybern.*, vol. 54, pp. 29-40, 1986.

[2] R. Eckhorn, R. Bauer, W. Jordan, M. Brosch, W. Kruse, M. Munk, and R. J. Reitboeck, "Coherent oscillations: A mechanism of feature linking in the visual cortex," *Biol. Cybern.*, vol. 60, pp. 121-130, 1988.

[3] C. M. Gray, P. Konig, A. K. Engel, and W. Singer, "Oscillatory responses in cat visual cortex exhibit inter-columnar synchronization which reflects global stimuli properties," *Nature*, vol. 338, pp. 334-337, 1989.

[4] Y. Yamaguchi and H. Shimizu, "Pattern recognition with figure-ground separation by generation of coherent oscillations," *Neural Networks*, vol. 7, pp. 49-63, 1994.

[5] Y. Hirakura, Y. Yamaguchi, H. Shimizu, and S. Nagai, "Dynamic linking among neural oscillators leads to flexible pattern recognition with figure-ground separation," *Neural Networks*, vol. 9, pp. 189-209, 1996.

[6] H. Sompolinsky, D. Golomb, and D. Kleinfeld, "Cooperative dynamics in visual processing," *Phys. Rev. A*, vol. 43, pp. 6990-7011, 1991.

[7] S. Kim and M. Y. Choi, "Arrays of resistively shunted Josephson junctions in magnetic fields," *Phys. Rev. B*, vol. 48, pp. 322-332, 1993.

[8] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence.* New York: Springer, 1984.

[9] D. Fisher, "Sliding charge-density waves as a dy-

namic critical phenomena," *Phys. Rev. B,* vol. 31, pp. 1396-1427, 1985.

[10] S. H. Strogatz, C. M. Marcus, R. M. Westervelt, and R. E. Mirollo, "Collective dynamics of coupled oscillators with random pinning," *Physica D,* vol. 36, pp. 23-50, 1989.

[11] R. L. Stratonovich, *Topics in the Theory of Random Noise.* New York: Gordon and Breach Science Publisher, 1967.

[12] Y. Kuramoto and I. Nishikawa, "Statistical macrodynamics of large dynamical systems: Case of a phase transition in oscillator communities," *J. Stat. Phys.,* vol. 49, pp. 569-605, 1987.

[13] H. Daido, "Scaling behavior as the onset of mutual entrainment in a population of interesting oscillators," *J. Phys. A,* vol. 20, pp. L629-L636, 1987.

[14] S. H. Strogatz and R. E. Mirollo, "Collective synchronization in lattices of non-linear oscillators with randomness," *J. Phys. A,* vol. 21, pp. L699-L705, 1988.

[15] H. Sakaguchi, S. Shinomoto, and Y. Kuramoto, "Local and global self-entrainments in oscillator lattices," *Prog. Theor. Phys.,* vol. 77, pp. 1005-1010, 1987.

[16] D. Golomb, D. Hansel, B. Shraiman, and H. Sompolinsky, "Clustering in globally coupled phase oscillators," *Phys. Rev. A,* vol. 45, pp. 3516-3530, 1992.

[17] K. Okuda, "Variety and generality of clustering in globally coupled oscillators," *Physica D,* vol. 63, pp. 424-436, 1993.

[18] S. Shinomoto and Y. Kuramoto, "Phase transitions in active rotator systems," *Prog. Theor. Phys.,* vol. 75, pp. 1105-1110, 1986.

[19] W. Horsthemke and R. Lefever, *Noise-Induced Transitions.* Berlin: Springer-Verlag, 1984.

[20] C. Van den Broeck, J. M. R. Parrondo, and R. Toral, "Noise-induced nonequilibrium phase transition," *Phys. Rev. Lett.,* vol. 73, pp. 3395-3398, 1994.

[21] S. H. Park and S. Kim, "Noise-induced phase transitions in globally coupled active rotators," *Phys. Rev. E,* vol. 53, pp. 3425-3430, 1996.

[22] S. H. Park, S. Kim, and S. K. Han, "Nonequilibrium phenomena in globally coupled phase oscillators: noise-induced bifurcations, clustering, and switching," *Int. J. Bifurcation and Chaos,* 1996, in press.

[23] S. Kim, S. H. Park, C. R. Doering, and C. S. Ryu, "Reentrant transitions in globally coupled active rotators with multiplicative and additive noises," *Phys. Lett. A,* 1996, submitted.

[24] C. R. Doering, "Modeling complex systems: stochastic processes, stochastic differential equations, and Fokker-Plank equations," in *1990 Complex Systems Summer School,* Natel and Stein, Ed., pp. 3-51, Addison Wesley, 1990.

[25] C. Kurrer and K. Schulten, "Effect of noise and perturbations on limit cycle systems," *Physica D,* vol. 50, pp. 311-320, 1991.

[26] H. Risken, *The Fokker-Planck Equation.* New York: Springer-Verlag, 1988.

[27] N. J. Newton, "Asymptotically efficient Runge-Kutta methods for a class of Itô and Stratonovich equations," *SIAM J. Appl. Math.,* vol. 51, pp. 542-567, 1991.

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