개선된 하중 및 변위 증분법

The Improved Load / Displacement Incremental Method

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ABSTRACT : 본 연구에서는 박벽 구조물의 기하학적 비선형 해석을 수행하기 위하여 개선된 하중 및 변위 증분의 조합법이 제시되었다. 제안된 알고리즘은 기존의 하중 및 변위 증분의 조합법이 고정된 증분량을 갖는 점을 개선하여, 수렴정도에 따라 증분량을 변화시킴으로써, 여러개의 임계점을 갖는 비선형 거동을 보다 효율적으로 추적하도록 하였다. 또한 하중 및 변위 증분법을 전환점을 첫 단계의 기울기에 비례한 값으로 대체함으로써 사용자의 편리를 도모하였다. 트러스, 공간 뼈대, 아치, 쉘 구조물 등의 기하학적 비선형 해석 예제를 통하여, 본 연구에서 제시한 개선된 하중 및 변위 증분의 조합법의 적용성을 입증하였다.

Keyword : 하중 및 변위 증분법, 후좌굴 해석, 박벽구조물

1. INTRODUCTION

In the post-buckling analyses of slender structures with rotational degrees of freedom, a special numerical technique must be adopted to trace the multiple load and displacement limit points, because the stiffness matrix in the vicinity of the limit point is nearly singular, and the descending branch of the load-deflection curve is characterized by a negative definite stiffness matrix.

Many methods have been proposed to solve limit point problems and there are a number of numerical algorithms such as the load incremental method[1], the displacement incremental method[2,3], and the arc-length method and its different variations[4~12]. The load incremental scheme based on the conventional Newton-Raphson method fails to converge in the case of passing the snap-through region with the load limit points. To circumvent this problem, Batoz and Dhatt[2] proposed the displacement incremental algorithm which preserved the symmetry and banded nature of

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the tangent stiffness matrix. This scheme can trace the snap-through behavior, but not the turning-back behavior with the displacement limit points.

In order to overcome these problems existing in the load and displacement incremental method, the arc-length method was proposed. This scheme keeps a arc-length constant. In recent years, considerable efforts have been devoted to the development of the reliable and efficient arc-length algorithms which allow various limit points to be traversed. Particularly, Bellini and Chulya[8] have critically studied the problem on the choice of arc-length increment and the choice of the appropriate root of the iterative load increment in the arc-length method. Kim and Chang[13] have proposed the automatic combination algorithm of load and displacement incremental method. This scheme keeps the load or displacement increment constant.

In this paper, the improved load/displacement incremental method that can trace the entire equilibrium path efficiently, is proposed. The so called '*tangent stiffness parameter*' is defined and a improved algorithm of combining the load and displacement incremental algorithms is proposed by using its geometric properties. With this algorithm, load and displacement limit points and bifurcation points with a steep negative slope are traceable. Numerical examples on the post-buckling behaviors of space trusses, space frames, arches and shell structures are presented to demonstrate the applicability of the proposed load/displacement incremental algorithm.

2. THE IMPROVED LOAD/ DISPLACEMENT INCREMENTAL METHOD

In this study, a improved algorithm of combining the load and displacement incremental method are presented to trace the non-linear behaviors of slender structures, efficiently. This scheme can trace the entire equilibrium path of a structure with multiple limit points by applying the displacement incremental method to the snap-through area and the load incremental method to the turning-back area alternately. Firstly, the load incremental method and the displacement incremental method are shortly presented, respectively, in order to develope the improved automatic load/displacement incremental method. Next the so called '*tangent stiffness parameter*' T_p is defined and finally, the technique of combining the load and displacement incremental algorithms is proposed by using this parameter.

2.1 Load Incremental Method

It is assumed that total displacements, and external and internal forces at the discrete time points 0, Δt, $2\Delta t, \cdots$, t have already been known. With the use of the Newton-Raphson iteration procedure and the assumption that the loading is proportional, the load incremental equilibrium equations between time t and $t+\Delta t$ are

$$ {}^{t+\Delta t}K_T^{(i-1)} \cdot \Delta U^{(i)} = {}^{t+\Delta t}\lambda P - {}^{t+\Delta t}F^{(i-1)} \qquad (1a)$$

$$ {}^{t+\Delta t}U^{(i)} = {}^{t+\Delta t}U^{(i-1)} + \Delta U^{(i)} \qquad i=1, 2, 3\cdots \qquad (1b)$$

$$ {}^{t+\Delta t}\lambda = {}^{t}\lambda + \Delta\lambda \qquad (1c)$$

The initial conditions :

$$ {}^{t+\Delta t}U^{(0)} = {}^{t}U, \ {}^{t+\Delta t}F^{(0)} = {}^{t}F \qquad (1d)$$

where ${}^{t+\Delta t}K_T^{(i-1)}$ denotes the tangent stiffness matrix consisting of the linear and the nonlinear stiffness matrices in the ith iteration, $\Delta U^{(i)}$ is an incremental displacement vector, λ is the load parameter, P is the reference load vector denoting the relative ratio of the externally applied load vector, ${}^{t+\Delta t}\lambda P$ is the nodal force vector in the configuration at time $t+\Delta t$, and ${}^{t}F$ is the nodal force vector corresponding to the element stresses.

In the load incremental method, increment of load parameter $\Delta\lambda$ remains constant during iteration process. Therefore, the total number n of equilibrium equations becomes equal to that of the unknown displacement components. The load incremental algorithm is summarized as follows :

1) Form equation (1a), the displacement increment $\Delta U^{(i)}$ due to the unbalanced load ${}^{t+\Delta t}\lambda P - {}^{t+\Delta t}F^{(i-1)}$ is computed in the ith iteration.

2) The total displacement ${}^{t+\Delta t}U^{(i)}$ is evaluated, and the internal nodal force ${}^{t+\Delta t}F^{(i)}$ corresponding to the total displacement is determined.

3) The new unbalanced load vector ${}^{t+\Delta t}\lambda P - {}^{t+\Delta t}F^{(i)}$ is evaluated by keeping the load parameter ${}^{t+\Delta t}\lambda$ constant, and the Newton-Raphson method is iteratively applied until the equilibrium condition at time $t+\Delta t$ is satisfied.

4) If the equilibrium state is reached, a k-th incremental displacement component $\Delta U_{,k}$ $(= {}^{t+\Delta t}U_{,k} - {}^{t}U_{,k})$ is stored. This value is used in order to switch the load incremental scheme to the displacement scheme auto-matically, when the displacement incremental method should be applied in next step.

2.2 Displacement Incremental Method

In the displacement incremental algorithm, it is assumed that the increment of the load parameter is unknown but the increment of a specific displacement component is kept constant during the iteration. Using the Batoz and Dhatt technique[2] based on the above assumption, equation (1) may be rewritten as

$$ {}^{t+\Delta t}K_T^{(i-1)} \cdot \Delta U^{(i)} = \Delta\lambda^{(i)} \cdot P - R^{(i)} \quad (2a)$$

$$ R^{(i)} = {}^{t+\Delta t}\lambda^{(i-1)} \cdot P - {}^{t+\Delta t}F^{(i-1)}, \quad i=1, 2, 3 \cdots\cdots \quad (2b)$$

$$ {}^{t+\Delta t}U^{(i)} = {}^{t+\Delta t}U^{(i-1)} + \Delta U^{(i)} \quad (2c)$$

$$ {}^{t+\Delta t}\lambda^{(i)} = {}^{t+\Delta t}\lambda^{(i-1)} + \Delta\lambda^{(i)} \quad (2d)$$

The initial conditions :

$$ {}^{t+\Delta t}U^{(0)} = {}^{t}U, \ {}^{t+\Delta t}F^{(0)} = {}^{t}F, \ {}^{t+\Delta t}\lambda^{(0)} = {}^{t+\Delta t}\lambda \quad (2e)$$

where $\Delta\lambda^{(i)}$ is the incremental load parameter and $R^{(i)}$ is the unbalanced load vector in the ith iteration.

Since the total number n of equations (2a) and (2b) is less than the number $n+1$ of unknowns, a constraint equation to keep the increment of a specific displacement component constant during the iteration is introduced. In the displacement incremental algorithm, the increments of the load parameter and the displacement vector are calculated using this constraint equation. The displacement incremental algorithm to evaluate $\Delta\lambda^{(i)}$ and $\Delta U^{(i)}$ is summarized as follows :

1) The displacement increments $\Delta U_R^{(i)}$ and $\Delta U_P^{(i)}$ due to the unbalanced load $R^{(i)}$ and the reference load P, respectively, are computed from equation (3).

$$ {}^{t+\Delta t}K_T^{(i-1)} \cdot \Delta U_R^{(i)} = \mathrm{R}^{(i)} \quad (3a)$$

$$ {}^{t+\Delta t}K_T^{(i-1)} \cdot \Delta U_P^{(i)} = \mathrm{P} \quad (3b)$$

2) The constraint equation for the k-th component of the displacement vector in the first iteration is

$$\Delta U_{,k}^{(1)}=\Delta\lambda^{(1)}\cdot\Delta U_{P,k}^{(1)}-\Delta U_{R,k}^{(1)}=(\Delta U_{,k})_{new} \quad (4)$$

where $\Delta U_{,k}$ denotes the k-th component of vector ΔU. The increment of the load parameter in the first iteration is calculated from equation (4) and the remaining incremental displacement components are determined from equation (5).

$$\Delta U^{(1)}=\Delta\lambda^{(1)}\cdot\Delta U_{P}^{(1)}-\Delta U_{R}^{(1)} \quad (5)$$

3) The constraint equation for the k-th displacement component after the first iteration is

$$\Delta U_{,k}^{(i)}=\Delta\lambda^{(i)}\cdot\Delta U_{P,k}^{(i)}-\Delta U_{R,k}^{(i)}=0 \quad (6)$$

From equation (6), increment of the load parameter is determined and the remaining incremental displacement components are obtained from equation (7).

$$\Delta U^{(i)}=\Delta\lambda^{(i)}\cdot\Delta U_{P}^{(i)}-\Delta U_{R}^{(i)} \qquad i=1,2,3\cdots \quad (7)$$

4) The total displacement is evaluated from the incremental displacement, and the new unbalanced load vector ${}^{t+\Delta t}\lambda^{(i)}P-{}^{t+\Delta t}F^{(i)}$ corresponding to the load parameter ${}^{t+\Delta t}\lambda^{(i)}$ is determined.

5) If the equilibrium state is reached, a incremental load parameter $\Delta\lambda(={}^{t+\Delta t}\lambda-{}^{t}\lambda)$ is stored. This value is used in order to switch the displacement incremental scheme to the load scheme automatically, when the load incremental method should be applied in next step.

2.3 Improved Combination Algorithm of the Load and Displacement Incremental Method

In order to introduce the key idea of the combined load / displacement incremental method, the relation curve between a load parameter and a specific displacement(see Fig. 1) is considered. In Figure 1, the load incremental method can find the turning-back regions(solid lines) with the displacement limit points (points C, D) but not the snap-through regions(dot lines) with the load limit points(points B, E), whereas the displacement incremental method can trace the snap-through path but not the turning-back path. Therefore, the entire non-linear path with multiple limit points can be traced by applying the displacement incremental method to the snap-through path and the load incremental method to the turning-back path alternately.

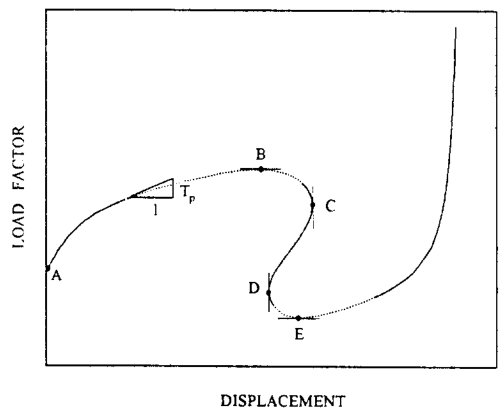


Fig. 1 Load - Displacement Curve in the Case of Snap-buckling

For efficient combination of two incremental schemes and automatic generation of the load and displacement increments, three important problems should be solved at the beginning of each time step :

1. the criterion to choose the load or displacement incremental method
2. the determination of the sign of the load or displacement increment.
3. the determination of the magnitude of the load or displacement increment.

Firstly, to determine the type of the incremental method applied in each time step, the '*tangent stiffness parameter*' T_p is defined as follows(see Fig. 1)

$$T_p = \frac{d\lambda}{dU_{,k}} \tag{8}$$

where $d\lambda$ and $dU_{,k}$ are the incremental load parameter and the increment of the specific displacement component, respectively. The parameter T_p is calculated by inverting a specific component of the displacement increment due to the reference load vector in the first iteration of each time step. Since T_p means the tangent of the relation curve between the load parameter and the specific displacement component, the absolute value of T_p becomes small in the snap-through area and large in the turning-back area. Therefore, the type of the incremental method can be automatically determined by using this geometric property of T_p. Namely, the load incremental method is applied if the absolute value of T_p is greater than the inputted positive value. Otherwise, the displacement incremental method is applied.

Secondly, the sign of the current increment, *i. e.* the load or displacement increment in the current time step, should be determined. In Figure 1, it can be noted that the specific displacement increment has the same sign in the snap-through area, and the load increment the same sign in the turning-back area. Accordingly, regardless of the type of the previous incremental scheme, the sign of the current increment should be the same as that in the previous time step.

Thirdly, the magnitude of the current increment should be determined. In this study, two schemes generating the current increment automatically are presented to ensure the efficiency of the load/displacement incremental method. The first scheme varies the length of the load or displacement increment according to the square root of the convergent ratio in the previous step like as the arc-length algorithm[8] (see equation (10) and (12)). Whereas, the second scheme[13] keeps constant the absolute value of the load increment or a specific displacement increment (see equation (11) and (13)).

Basing on the above procedures, the entire non-linear path with multiple limit points can be traced by applying the displacement incremental method to the snap-through path and the load incremental method to the turning-back path. Consequently, the combination scheme of the load and the displacement incremental method is summarized as follows :

1) Initialize state variables such as displacement, stress and strain components, reactions and unbalanced load and prepare input parameters such as geometric and material properties, connectivity, and boundary conditions.

2) Prepare the reference load vector P where

represents only the relative ratio of load components.

3) Start the time step loop

4) Evaluate the tangent stiffness matrix and the displacement increment due to the reference load vector before iteration process of each time step.

5) Evaluate T_p as the inverse of the specific component of the displacement increment due to the reference load vector.

6) Apply the incremental displacement method in the current time step if the following condition (9) is satisfied. Otherwise, apply the incremental load method.

$$|{}^{t}T_p| \leq DETOL \cdot |{}^{o}T_p| \tag{9}$$

where ${}^{o}T_p$ and ${}^{t}T_p$ denote the tangent stiffness parameters at the initial and current time step, respectively, and $DETOL$ is the positive ratio given by the input value (usually, $DETOL = 0.3 \sim 0.4$).

7) In the first time step, the magnitude and sign of the load or displacement increment is given by the input value.

8) From the step 6), if the scheme of the current time step is determined as the load incremental method, the magnitude and sign of the load increment is obtained by the following two procedures :

(a) scheme 1 : the variable length scheme[8]

$$(\Delta\lambda)_{new} = AMP \cdot (\Delta\lambda)_{old} \tag{10a}$$

$$AMP = \sqrt{\frac{NTARG}{NCONV}} \tag{10b}$$

where $(\Delta\lambda)_{new}$ and $(\Delta\lambda)_{old}$ are the increments of the load parameter in the current step and the previous step, respectively. $NCONV$ is the converged iteration number in the previous step, and $NTARG$ is the target iteration number given by the input value.

(b) scheme 2 : the constant length scheme[13]

$$(\Delta\lambda)_{new} = \frac{(\Delta\lambda)_{old}}{|(\Delta\lambda)_{old}|} \cdot (\Delta\lambda)_{inp} \tag{11}$$

where $(\Delta\lambda)_{inp}$ is the positive input value for the load parameter increment.

9) From the step 6), if the scheme of the current time step is determined as the displacement incremental method, the magnitude and sign of the displacement increment is obtained by the following procedure :

(a) scheme 1 : the variable length scheme[8]

$$(\Delta U_{,k})_{new} = AMP \cdot (\Delta U_{,k})_{old} \tag{12}$$

where $(\Delta U_{,k})_{new}$ and $(\Delta U_{,k})_{old}$ are the displacement increments in the current step and the previous step, respectively.

(b) scheme 2: the constant length scheme[13]

$$(\Delta U_{,k})_{new} = \frac{(\Delta U_{,k})_{old}}{|(\Delta U_{,k})_{new}|} \cdot (\Delta U_{,k})_{inp} \tag{13}$$

where $(\Delta U_{,k})_{inp}$ is the positive input value for the displacement increment.

10) With the load or displacement incremental method, apply the iterative Newton-Raphson method within admissible maximum iteration ($MITER$) until the following convergence condition is satisfied :

$$|{}^{t+\Delta t}\lambda^{(i)}P - {}^{t+\Delta t}F^{(i)}| \leq 100 \cdot TOLER \cdot |{}^{t}\lambda P| \tag{14}$$

where $TOLER$ is the convergence tolerance given by the input value.

11) If it is converged within admissible maximum iteration, store the converged state variables for the current configuration and go to the step 4).

12) In order to avoid a divergence, if not converged within maximum iteration, reread the stored state variables at the previous step and with half-increment of the load or the specific displacement component, go to the step 4) and the iteration is restarted from the previous equilibrium state.

From the proposed algorithm, it can be noted that in the first time step, the load or displacement increment should be given as the inputted value, but in the subsequent steps, the sign and the magnitude of the current increment is automatically generated.

3. NUMERICAL EXAMPLES

In this chapter, numerical examples focused on geometrically non-linear analysis of slender structures using the space truss element, the space Hermitian frame element[1], and the curved beam[1] and shell element[14] are presented to demonstrate the reliability and the computational efficiency of the proposed algorithm. The space truss element means the two noded straight element with only translational degrees of freedom and the space frame element denotes the two noded element using Hermitian polynomials as the shape function. Also, the curved beam and shell elements are the degenerate 3-node beam[1] and 9-node shell elements[14]. In the subsequent examples, the geometric and physical properties as well as the control parameters of each problem are given in the corresponding Figures.

3.1 Star-Shaped Shallow Dome Truss

Figure 2 shows a hinged shallow dome truss subjected to a vertical point load($P=f\ p$) at the apex and composed of thirty truss members. Here, f and p are the load incremental factor and the reference load (=50), respectively. This dome has been analysed by various authors[7,8,15] to trace the post-buckling load-deflection path for various loading conditions since it is a highly non-linear problem with multiple limit points.

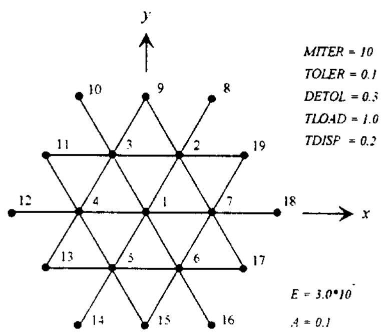


Fig. 2 Star-shaped Shallow Dome Truss under a Center Load

In Figure 3, the analyzed results of the load versus the displacements of joint 1 are compared with those of Bellini and Chulya[8] using a arc-length algorithm. The two results appear to be in complete agreement. Also, the limit loads obtained by this study and Kwok *et al.*[15] are displayed in Table 1. This shows that the results of this study agree well with those of Kwok *et al.* using model trust region quasi-Newton and tunnelling method. The post-buckling load-displacement paths of joint 7 are presented in Figure 4.

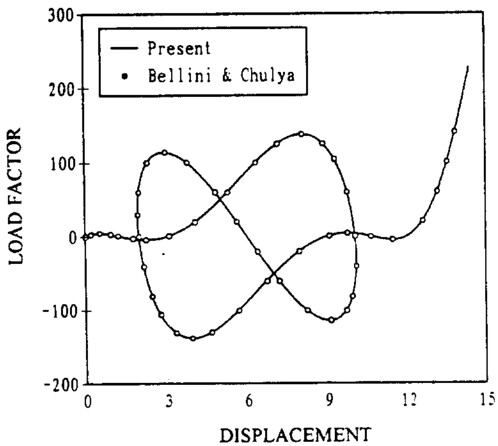


Fig. 3 Load - Vertical Displacement Curves of Joint 1

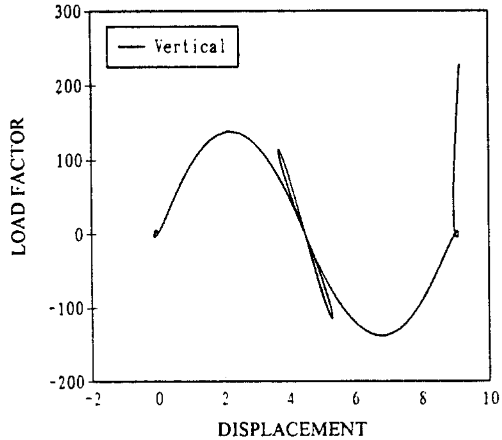
Table 1. Limit Loads of Shallow Dome Truss

Limit Loads	Kwok *et al.*[15] (10^6 P/EA)	This Study (10^6 P/EA)
Third Limit Load	2297.55	2299.73
Fourth Limit Load	-1906.60	-1905.00
Fifth Limit Load	1887.40	1903.37
Sixth Limit Load	-2297.65	-2299.22

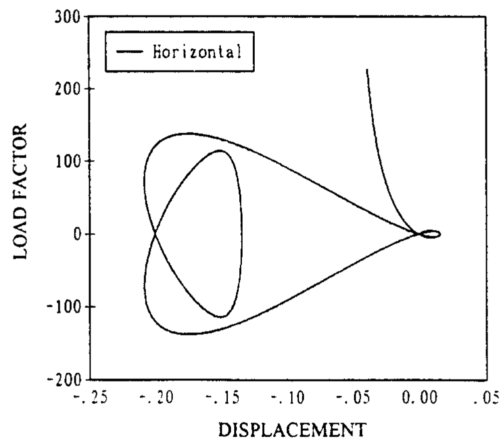
3.2 Hexagonal Space Frame

This problem consists of a three dimensional frame composed of twelve members, with half of them laid out as an hexagon, and the other half making up the diagonals of the hexagon. The load is applied vertically on the central node. Each member of the frame is modeled by two degenerate beam elements[1].

The evolution of the deflection of the apex, while the load is varied, is given in Figure 6. This represents the typical non-linear behavior with two load limit points. Also, Figure 6 shows that the results of this study agree well with those of Papadrakakis[16] using the two vector iteration methods.



(a) Load - Vertical Displacement Curve



(b) Load - Horizontal Displacement Curve

Fig. 4 Load - Displacement Curve of Joint 7

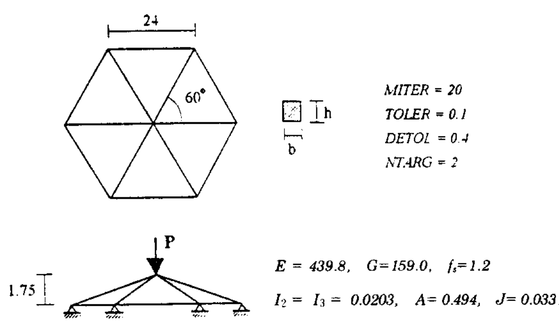


Fig. 5 Hexagonal Space Frame under a Center Load

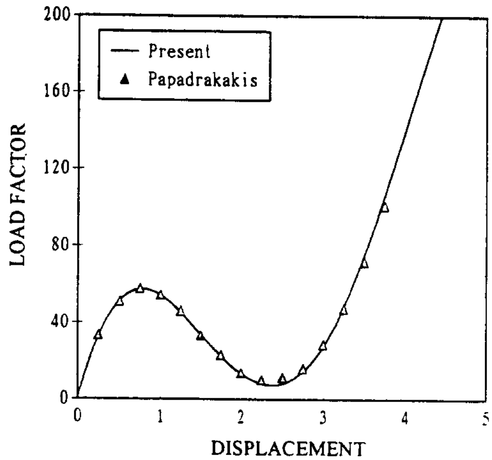


Fig. 6 Load - Vertical Displacement Curves at Apex

3.3 Hinged Right-Angled Frame

Figure 7 shows a hinged right-angled frame subjected to a force($P=f\ p$) at the distance L from the left hinge. Here, f and p are the load incremental factor and the reference load, respectively. The frame is modeled using twenty Hermitian beam elements and analyzed using *the constant length scheme* of the load and the displacement increment(see equation (11) and (13)).

Figure 8 displays the analyzed results of the load parameter versus the horizontal(D_x) and

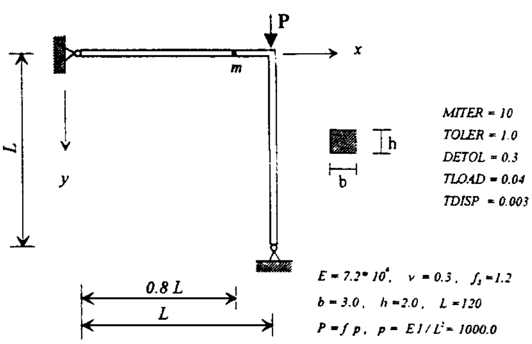


Fig. 7 Hinged Right-Angled Frame under a Point Load at $x = L$

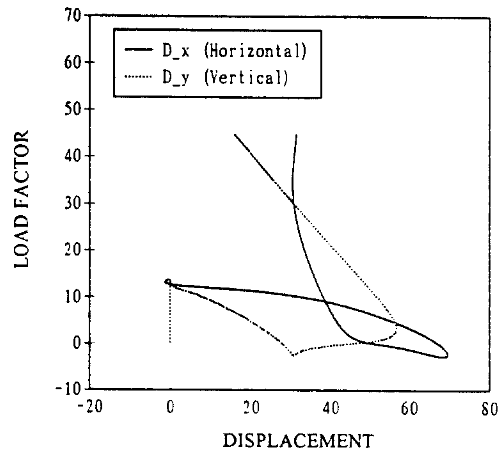


Fig. 8 Load-Displacement Curves of Point *m*

vertical displacements(D_y) at the point *m*. This shows that the frame buckles to the clockwise at the upper limit load f_u=13.8, which is in good agreement with the analytic buckling load of Kouhia and Mikkola[9] (f_{cr}=13.9).

3.4 Hinged Cylindrical Shell

This is an example to study the geometrically non-linear behavior of a shallow cylindri-

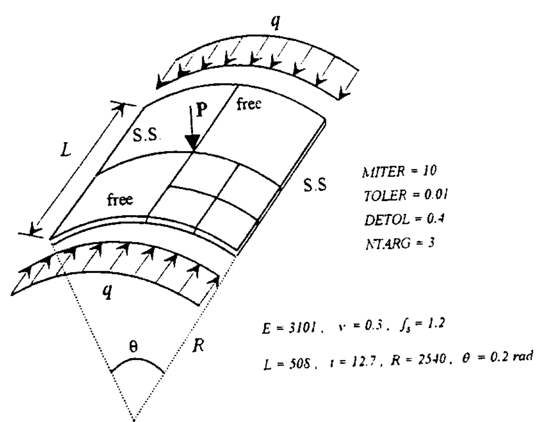


Fig. 9 Hinged Cylindrical Shell Under a Center Load and Uniform Compression

cal shell under a concentrated load and the uniform axial compression. The shell shown in Figure 9 is hinged at the longitudinal edges and is free along the curved boundaries. Because of the symmetry, only a quarter of the shell is modeled using four degenerate shell elements[14]. To investigate the effect of axial compression on the buckling load, the shell is analyzed for the three cases of axial compression ; *i. e.* $q=0, 5, 10$ N/mm.

Figure 10 displays the plots between the load parameter and the vertical displacement at the center of shell. These non-linear equilibrium paths exhibit the snap-through as well as the turning-back phenomena, with horizontal and vertical tangents. This shows that the first limit load decreases as the axial compression increases. In the case of no axial compression, the results of present study is compared with those of Ramm[5] in Figure 10.

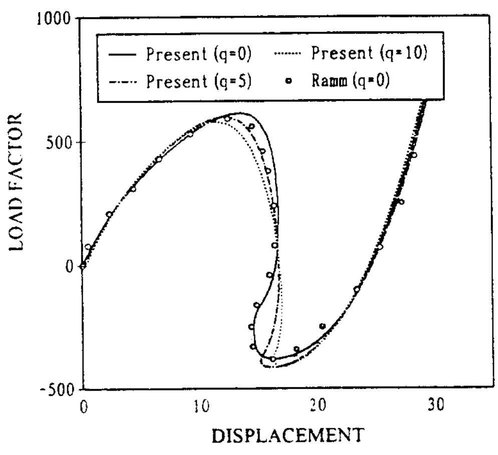


Fig. 10 Load -Vertical Displacement Curves at the Center of Shell

3.5 Hinged Shallow Circular Arch

Figure 11 shows a schematic representation of a hinged circular arch subjected to a eccentric point load. The arch is modeled by eleven Hermitian beam elements.

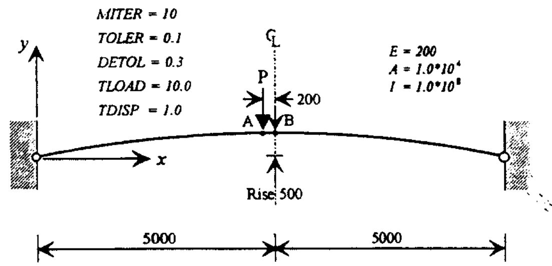
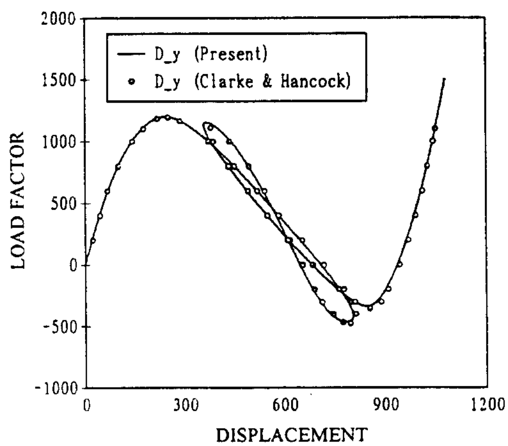
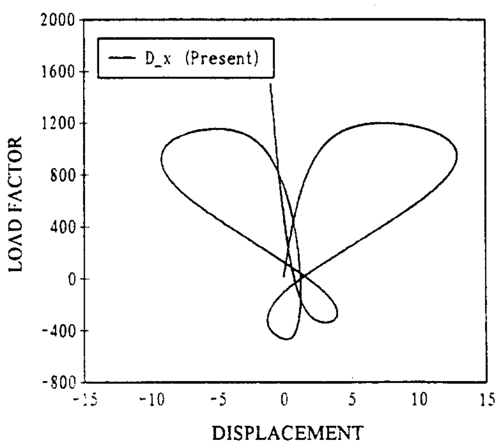


Fig. 11 Hinged Shallow Chricular Arch under a Eccentric Load



(a) Load - Vertical Displacement Curve



(b) Load - Horizontal Displacement Curve

Fig. 12 Load-Displacement Curves of Pont A

In Figure 12, the results obtained by the present study are compared to those by Clarke and Hancock[11] with the load-displacement curve of the point A. This shows that the results of the present study agree well with those of Clarke and Hancock. Resultantly, it is judged that the present load/displacement incremental method can trace the full equilibrium path of the arch which has four snap-through and two turning-back regions.

4. CONCLUSIONS

The improved automatic combination algorithm of the load and displacement method for the geometrically non-linear analysis has been presented. Post-buckling behaviors of the plane and space frames, arches, and shell structures have been analyzed using the space truss element, the space Hermitian frame element, and the degenerate beam and shell elements. Through numerical examples focused on the post-buckling analyses of slender structures, finite element solutions are compared with various researcher's results. Resultantly, it is judged that the proposed load/displacement incremental algorithm can accurately trace the entire postbuckling equilibrium path of the slender structures with multiple limit points, efficiently.

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