On Testing Monotonicity of Mean Residual Life from Randomly Censored Data

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ABSTRACT

This paper proposes a new nonparametric test for testing the null hypothesis that the MRL is constant against the alternative hypothesis that the MRL is decreasing (increasing) for ramdomly censored data. The proposed test statistic is a L-statistic, and we use L-statistic theory to establish its asymptotic normality of the test statistic. We discuss the efficiency loss due to censoring and also calculate the asymptotic relative efficiencies of our test statistic with respect to the Chen, Hollander and Langberg's test for several alternatives.

I. INTRODUCTION

Let $F(\cdot)$ be a continuous life distribution (i.e. $F(t) = 0$ for all $t < 0$) with survival function $\bar{F}(\cdot) = 1 - F(\cdot)$, F(0)=0 and finite mean $\mu = \int_0^\infty \bar{F}(x)dx$. Then the corresponding mean residual life (MRL) function is defined for $t > 0$ by

$$m(t) = \begin{cases} \int_t^\infty \bar{F}(u)du/\bar{F}(t) & \text{if } \bar{F}(t) > 0 \\ 0 & \text{if } \bar{F}(t) = 0. \end{cases} \quad (1)$$

The MRL function, $m(t)$, represents the expected remaining life of an item at age t.

The decreasing mean residual life (DMRL) class is introduced by Bryson and Siddiqui [1] as a criterion for aging. A life distribution F is said to be in DMRL class if

$$m(s) \geq m(t) \quad for \;\; all \;\; 0 \leq s \leq t. \quad (2)$$

The dual class, increasing mean residual life (IMRL) class, is defined by reversing the inequality in the definition of DMRL class.

The MRL function has attracted a great deal of interest among reliability analysts, biometrists, and actuarial scientists. Guess and Proschan [2] give an excellent discussion for MRL function.

Hollander and Proschan [3] suggest a procedure for testing exponentiality against the alternative of monotone and nonconstant MRL. Chen, Hollander and Langberg [4] extend Hollander and Proschan's test to the case of randomly censored data. Bergman and Klesfjo [5] develop a family of test statistics intended for testing exponentiality against DMRL when the data is both complete and censored. Aly [6] suggests a set of tests for testing monotonicity of MRL one of which deals with detecting DMRL. Ahmad [7] proposes a new test procedure for testing exponentiality against DMRL alternatives and shows that his test performs better than Hollander and Proschan's test by calculating Pitman asymptotic relative efficiency for several alternatives. Lim and Park [8] generalize Ahmad's test to accomodate the situation that the data is randomly censored and compare their test with Chen, Hollander and Langberg's [4] test.

In this paper, we propose a new test statistic for testing the monotonicity of MRL of F when the data is randomly censored. Since the test statistic is a L-statistic, we use the L-statistic theory to establish the asymptotic normality of our test statistic. We discuss the efficiency loss due to censoring and also calculate the asymptotic relative efficiencies of our test statistic with respect to Chen, Hollander and Langberg [4] for several alternatives.

II. TEST FOR DMRL ALTERNATIVES

In this section, we derive test statistics for testing

H_0 : F is a constant MRL distribution.
That is, $F(t) = 1 - exp(-t/\mu)$, $\mu > 0, t \geq 0$.

against

H_1 : F is DMRL (and not constant MRL) distribution,

or

H_2 : F is IMRL (and not constant MRL) distribution,

based on a random sample $X_1, X_2, \ldots, X_n$ from a continuous life distribution F and a random sample $Y_1, Y_2, \ldots, Y_n$ from a continuous censoring distribution H. The censoring distribution H is assumed to be unknown and is treated as a nuisance parameter. We also assume that X's and Y's are mutually independent.

Aly [6] proposes a measure of DMRL-ness which is based on the first derivative of the mean residual life. He considers the parameter

$$\gamma(F) = \int_0^\infty \bar{F}(t)(1 + ln\bar{F}(t))dt. \quad (3)$$

It is easy to see that $\gamma(F) = 0$ under H_0, and $\gamma(F) \geq 0$ under H_1. Simple algebra shows, with an additional assumption $\lim_{t\to\infty} t\bar{F}(t)\, ln\,\bar{F}(t) = 0$, that $\gamma(F)$ can be rewritten as

$$\begin{aligned}\gamma(F) &= \int_0^\infty [\bar{F}(t)ln\bar{F}(t) + \bar{F}(t)]dt \\ &= \int_0^\infty t[2 + ln\bar{F}(t)]dF(t) \\ &= \int_0^\infty tJ(F(t))dF(t), \quad (4)\end{aligned}$$

where $J(x) = 2 + ln(1-x)$.

Aly [6] obtains the test statistic, $L_n = \gamma(F_n)$, by replacing F in $\gamma(F)$ by the empirical distribution F_n. In randomly censored model, we use Kaplan-Meier estimator F_n^* of F, instead of F_n (cf. Kaplan and Meier [9]).

In the censoring model, instead of observing $X_1, X_2, \ldots, X_n$, we observe the pairs (Z_i, δ_i), i=1,2,...,n, where $Z_i = min(X_i, Y_i)$ and $\delta_i = 1$ if $Z_i = X_i$, $\delta_i = 0$ if $Z_i = Y_i$. That is, $Z_1, Z_2, \ldots, Z_n$ are observations from a life distribution with survival function $\bar{K}(t) = \bar{H}(t)\bar{F}(t)$.

Kaplan and Meier's [9] the product-limit estimator is

$$\bar{F}_n^*(t) = \prod_{\{i: Z_{(i)} \leq t\}} [(n-i)(n-i+1)^{-1}]^{\delta_{(i)}}, \quad t \in [0, Z_{(n)}] \quad (5)$$

where $Z_{(0)} = 0 < Z_{(1)} < \cdots < Z_{(n)}$ denote the ordered Z's and $\delta_{(i)}$ is the δ_i corresponding to $Z_{(i)}$. We use the standard conventions:

1. $Z_{(n)}$ is treated as an uncensored observation even though it is censored.

2. When censored observations are tied with uncensored observations, the uncensored observations are treated as preceeding the censored observations.

By replacing $\bar{F}$ in $\gamma(F)$ by the Kaplan-Meier estimator, the proposed test statistic is :

$$\begin{aligned}L_n^c &\stackrel{def}{=} \hat{\mu}^{-1}\gamma(F_n^*) = \hat{\mu}^{-1}\int_0^\infty \bar{F}_n^*(x)(1 + ln\bar{F}_n^*(x))dx \\ &= \hat{\mu}^{-1}\sum_{i=1}^n [\prod_{j=1}^{i-1} c_j^{\delta_{(j)}}][ln(\prod_{j=1}^{i-1} c_j^{\delta_{(j)}} + 1)] \\ &\quad (Z_{(i)} - Z_{(i-1)}), \quad (6)\end{aligned}$$

where $c_j = \frac{n-j}{n-j+1}$ for j=1,2,...,n and $\hat{\mu} = \sum_{i=1}^n \{\prod_{j=1}^{i-1} c_j^{\delta_{(i)}}\}(Z_{(i)} - Z_{(i-1)})$.

To obtain the limiting distribution of our test statistic, we assume the following conditions on H and F.

A1: For some $\beta \in (0, \frac{1}{2})$,

$$\int_0^\infty \bar{F}^\beta(x)dx < \infty$$

and

$$\int_0^\infty [\bar{F}^{2\beta}(x)\bar{H}(x)]^{-1}dF(x)dx < \infty$$

A2: $\sqrt{n}\int_{M_n}^\infty \bar{F}(x)dx \xrightarrow{p} 0,$

where $M_n = \max(Z_1, \ldots Z_n)$.

We also assume $\sigma^2_{J^*}(F, H) > 0$, where $\sigma^2_{J^*}(F, H)$ is the asymptotic variance. An excellent interpretation of assumptions A1 and A2 is given by Guess [10]. Applying Theorem 4.1 in Joe [11] and the Slutsky's Theorem (cf. Serfling [12]), we have

$$\sqrt{n}(L_n^c - \gamma(F)/\mu) \xrightarrow{d} N(0, \sigma^2), \quad (7)$$

where

$$\sigma^2 = \sigma^2_{J^*}(F, H)/\mu^2, \quad (8)$$

$$J^*(u) = J(u) - \gamma(F)/\mu, \quad (9)$$

$$\sigma^2_{J^*}(F, H) = \int\int \bar{F}(x)\bar{F}(y)J^*(F(x))J^*(F(y)) \left(\int_0^{x\wedge y}[\bar{K}(u)\bar{F}(u)]^{-1}dF(u)\right)dxdy, \quad (10)$$

$$\bar{K}(u) = \bar{H}(u)\bar{F}(u) \quad (11)$$

and $x \wedge y$ = min(x,y).

In particular, under H_0, we obtain $\gamma(F) = 0$ and $J^*(u) = J(u)$. Straightforward calculations show that the asymptotic null variance reduces to

$$\sigma^2_J(F, H) = \int_0^1 g(z)[\bar{K}(-\mu lnz)]^{-1}dz, \quad (12)$$

where $g(z) = z + 2zlnz + z(lnz)^2$. In the case of no censoring, $\sigma^2_J(F, H) = 1$ agreeing with the asymptotic null variance for the complete data.

A consistent estimator, $\hat{\sigma}^2_J(F, H)$, of $\sigma^2_J(F, H)$ is obtained by replacing $\bar{K}$ in (12) with $\hat{\bar{K}}$, where $\hat{\bar{K}} = n^{-1}\sum_{i=1}^n I(Z_i - t)$ is an empirical survival function of the Z values. For the purpose of computation, it is convenient to write

$$\hat{\sigma}^2_J(F, H) = \frac{1}{4} + \left\{\sum_{i=1}^{n-1}\frac{n}{(n-i+1)(n-i)}\left[\frac{1}{4} - \frac{1}{2}Z_{(i)}\hat{\mu}^{-1} + \frac{1}{2}Z^2_{(i)}\hat{\mu}^{-2}\right]B_i(4)\right\} - n\left[\frac{1}{4} - \frac{1}{2}Z_{(n)}\hat{\mu}^{-1} + \frac{1}{2}Z^2_{(n)}\hat{\mu}^{-2}\right]B_n(4), \quad (13)$$

where $B_i(a) = exp[-aZ_{(i)}/\hat{\mu}]$.

The proposed test at approximate α level of significance is to reject H_0 in favour of DMRL (IMRL) alternatives if $\sqrt{n}L_n^c/\hat{\sigma}_J(F, H) \geq z_\alpha(\leq -z_\alpha)$, where z_α is the upper α-percentile of a standard normal distribution.

III. EFFICIENCY LOSS DUE TO CENSORING

In this section, we study the efficiency loss due to censoring by computing the efficiency of Aly [6] test based on L_n for uncensored model and the efficiency of our test based on L_n^c for censored model. Since L_n and L_n^c have the same asymptotic means, the Pitman asymptotic relative efficiency (ARE) of the test based on L_n^c with respect to the test based on L_n is given by

$$e_H(L_n^c, L_n) = \frac{1}{\sigma^2_J(F, H)}. \quad (14)$$

Note that $e_H(L_n^c, L_n)$ is independent of the distribution family in the alternative hypo-

Table 1. Efficiency loss when the censoring distribution is exponential with parameter λ.

λ	1/2	1/3	1/4	1/5	1/10	1/30	1/50	1/70	1/100
Efficiency Loss	.100	.267	.397	.492	.722	.902	.941	.958	.970

thesis but depends on the censoring distribution H. e_H can be interpreted in the following manner. Suppose that we are testing DMRL (IMRL) alternatives which are closed to exponential distribution. If we want the same power when using L_n^c as for L_n, we need n/e_H observations if L_n-test is based on n observations.

We consider the case when the censoring distribution is an exponential distribution, H(x)=1-exp[-λx], x $\geq$ 0 with restriction $\lambda < 1$. Then we obtain (with $\mu = 1$)

$$e_H(L_n^c, L_n) = \frac{(1-\lambda)^3}{\lambda^2+1} \qquad (15)$$

The values in Table 1 show that the value of e_H increases to 1 as λ decreases. It is clear from the formula that the efficiecy loss tends to 0 as λ (the amount of censoring) is getting small.

IV. ASYMPTOTIC RELATIVE EFFICIENCY

Chen, Hollader and Langberg [4] suggest a test procedure designed for testing H_0 against H_1 when the sample is randomly censored. In this section, we compare our tests based on L_n^c with respect to Chen, Hollander and Langberg's [4] test based on V^c. The Pitman ARE of the test based on L_n^c with respect to one based on V^c is given by

$$e_F(L_n^c, V^c) = \left(\frac{m_1'(\theta_0)}{m_2'(\theta_0)} \frac{\xi_0}{\tau_0}\right)^2, \qquad (16)$$

where τ_0^2 and ξ_0^2 are asymptotic null variances of L_n^c and V^c, respectively, $m_1'(\theta_0)$ and $m_2'(\theta_0)$ are the derivatives of $m_1(\theta)$ and $m_2(\theta)$ with respect to θ and evaluated at $\theta = \theta_0$ and $m_1(\theta)$ and $m_2(\theta)$ are the asymptotic means of L_n^c and V^c. That is,

$$m_1(\theta) = \int \bar{F}(x)dx + \int \bar{F}(x) ln \bar{F}(x)dx \qquad (17)$$

and

$$m_2(\theta) = -3^{-1}\int \bar{F}^4(x)dx + 2^{-1}\int \bar{F}^2(x)dx - 6^{-1}\int \bar{F}(x)dx. \qquad (18)$$

Let

$$A(\lambda) = \xi_0^2/\tau_0^2 = \frac{(1-\lambda)^{-3}}{36(1+\lambda^2)}\{(1-\lambda)^{-1} - 6(2-\lambda)^{-1} + 9(3-\lambda)^{-1} + 4(4-\lambda)^{-1} - 12(5-\lambda)^{-1} + 4(7-\lambda)^{-1}\}. \qquad (19)$$

By consideration of Pareto distribution, Makeham distribution and Weibull distribution, we obtain the following asymptotic relative efficiencies.

$$e_{F_1,H}(L_n^c, V^c) = 256A(\lambda) \qquad (20)$$

Table 2. Pitman ARE of the test based on L_n^c with respect to the test based on V^c for several alternative distributions.

Distribution	λ									
	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
Pareto	1.219	1.093	.952	.800	.644	.492	.348	.217	.109	.031
Makeham	1.071	.961	.836	.703	.566	.432	.305	.191	.095	.027
Weibull	1.427	1.280	1.114	.936	.754	.576	.407	.255	.127	.036

$$e_{F_2,H}(L_n^c, V^c) = 225A(\lambda) \qquad (21)$$

$$e_{F_3,H}(L_n^c, V^c) = \left(\frac{12}{ln2}\right)^2 A(\lambda). \qquad (22)$$

The values in Table 2 indicate that the test based on L_n^c compares favorably with the test based on V^c when the sample is slightly censored. In the case of no censoring, the values in the table are the same as the values in Aly [6].

For the purpose of illustration of our test, we analyze the survival data of 211 prostate cancer patients (cf. Bryson and Siddiqui [1]). Chen, Hollander and Langberg [4] and Lim and Park [8] apply their tests to the same data and conclude that this data strongly supports the DMRL alternative. Applying our test to the data, we obtain L_n^c=.396, $\hat{\sigma}$=.414 and thus $\sqrt{211}L_n^c/\hat{\sigma}$=8.75 whose associated p-value is less than .001. The test result strongly indicates that the survival time follows a decreasing mean residual life.

V. CONCLUSION

We have proposed a new nonparametric test for testing exponentiality against DMRL when the data is randomly censored. We use the L-statistic theory to establish the asymptotic normality of our test statistic. We obtain the efficiency loss due to censoring and also calulate the asympotic relative efficiencies of our test statistic with respect to Chen, Hollender and Langberg's [4] test for several alternatives. The values of the asymptotic relative efficiencies show that our test outperforms theirs when the data is slightly censored. A survival data on prostate cancer patients is considered in order to illustrate our test.

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