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DIMENSION OF THE LINK OF FACES ASSOCIATED WITH ANY SIMPLICIAL COMPLEX

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1. Introduction

Let Δ be a simplicial $(d-1)$-dimensional spherical surface with v vertices. In [Kl], Klee conjectured that the number $f_i(\Delta)$ of i-faces of Δ is less than or equal to a certain number $f_i(C(\Delta(v,d)))$ (upper bound conjecture, or UBC). In [St1], Stanley showed that the necessary condition of UBC is as follows:

"A commutative ring $k[\Delta]$ (so called Stanley-Reisner ring) associated with Δ is Cohen-Macaulay".

But in [Re], Reisner proved the following theorem:

A ring $k[\Delta]$ is Cohen-Macaulay if and only if Δ has Cohen-Macaulay links of vertices and $\tilde{H}_i(\Delta;k)=0, \quad i<\dim\Delta$.

If we calculate the dimension of link of faces associated with Δ, we can obtain another easier proof of the above theorem.

This paper devotes to calculate the dimension of $link_\Delta(\sigma)$ of a face σ associated with any d-dimensional simplicial complex Δ.

2. Preliminaries

Throughout this paper, $\mathbb{R}^N$ denote the N-dimensional Euclidean space.

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DEFINITION 2.1. Let W be a nonempty subset of $\mathbb{R}^N$. If there exist subspace U of $\mathbb{R}^N$ and $\alpha \in \mathbb{R}^N$ such that

$$W = U + \alpha := \{\mathbf{x} + \alpha | \quad \mathbf{x} \in U\}$$

then W is called an affine subspace of $\mathbb{R}^N$.

Let W be an affine subspace of $\mathbb{R}^N$. The dimension of W (denoted by dimW) is defined by the maximal number d such that there exist elements $\mathbf{w}_0, \mathbf{w}_1, \cdots, \mathbf{w}_d$ in W for which $\mathbf{w}_1 - \mathbf{w}_0, \cdots, \mathbf{w}_d - \mathbf{w}_0$ are lineary independent over $\mathbb{R}$.

Let X be a subset of $\mathbb{R}^N$ and α an element of X. If U is a subset of $\mathbb{R}^N$ spanned by $\{\mathbf{x} - \alpha | \mathbf{x} \in X\}$, then the affine subspace $U + \alpha$ (denoted by AFF(X)) is called an affine subspace spanned by X.

REMARK 2.2. (1) The definition of AFF(X) is independent of any choice of $\alpha \in X$.

(2) Any intersection of convex subsets of $\mathbb{R}^N$ is also a convex subset of $\mathbb{R}^N$. And so for a nonempty subset X of $\mathbb{R}^N$, there exists a smallest convex subset of $\mathbb{R}^N$ containing X. This set is called the convex closure of X which is denoted by CONV(X).

Then we have the well known lemma:

LEMMA 2.3. *Let* $X = \{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_\nu\} \subset \mathbb{R}^N$. *Then*

$$CONV(X) = \{\sum_{i=1}^{\nu} t_i \mathbf{x}_i | \quad 0 \le t_i \in \mathbb{R}, \quad \sum_{i=1}^{\nu} t_i = 1\}.$$

Proof. For convenience, we put

$$X' = \{\sum_{i=1}^{\nu} t_i \mathbf{x}_i | \quad 0 \le t_i \in \mathbb{R}, \quad \sum_{i=1}^{\nu} t_i = 1\}.$$

First, we will show, using induction on ν, that X' is contained in CONV(X). Suppose that

$$\sum_{i=1}^{\nu} t_i \mathbf{x}_i \in X', \quad where \quad 0 \le t_i \in \mathbb{R}, \quad \sum_{i=1}^{\nu} t_i = 1.$$

Then

$$\sum_{i=1}^{\nu} t_i \mathbf{x}_i = t_1 \mathbf{x}_1 + (1-t_1)\left[(\sum_{i=2}^{\nu} t_i \mathbf{x}_i)/(1-t_1) \right]$$
$$= t_1 \mathbf{x}_1 + (1-t_1)\left[\sum_{i=2}^{\nu} (t_i/(1-t_1))\mathbf{x}_i \right],$$

where $\sum_{i=2}^{\nu} t_i/(1-t_1) = (\sum_{i=2}^{\nu} t_i)/(1-t_1) = 1$. By inductive hypothesis, $\mathbf{x}'_1 := (\sum_{i=2}^{\nu} t_i \mathbf{x}_i)/(1-t_1) \in CONV(X)$. Thus $\sum_{i=1}^{\nu} t_i \mathbf{x}_i = t_1 \mathbf{x}_1 + (1-t_1)\mathbf{x}'_1 \in CONV(X)$.

Second, we will show that X' is a convex set. For

$$\alpha = \sum_{i=1}^{\nu} t_i \mathbf{x}_i, \quad \beta = \sum_{i=1}^{\nu} s_i \mathbf{x}_i \quad \in X' \quad and \quad 0 \le t \le 1,$$

we have

$$t\alpha + (1-t)\beta = t\sum_{i=1}^{\nu} t_i \mathbf{x}_i + (1-t)\sum_{i=1}^{\nu} s_i \mathbf{x}_i$$
$$= \sum_{i=1}^{\nu} (tt_i + (1-t)s_i)\mathbf{x}_i,$$

$$tt_i + (1-t)s_i \ge 0, \quad 1 \le i \le \nu.$$

But

$$\sum_{i=1}^{\nu} (tt_i + (1-t)s_i) = t\sum_{i=1}^{\nu} t_i + (1-t)\sum_{i=1}^{\nu} s_i = 1.$$

Thus $t\alpha + (1-t)\beta \in X'$. So X' is a convex set. Hence CONV(X) is contained in X'. This completes the proof. □

For a convex subset A of $\mathbb{R}^N$, the dimension of A is defined by

$$dimA = dimAFF(A).$$

A subset $\mathcal{P}$ of $\mathbb{R}^N$ is said to be a convex polytope if $\mathcal{P} =$ CONV(X) for some finite subset X of $\mathbb{R}^N$. The dimension of a convex polytope $\mathcal{P} \subset \mathbb{R}^N$ (denoted by $dim\mathcal{P}$) is defined by the dimension of $\mathcal{P}$ as a convex set.

Let $\mathcal{P} \subset \mathbb{R}^N$ be a convex polytope. A face $\mathcal{F}$ of $\mathcal{P}$ is called an i-face if $dim\mathcal{F} = i$. In particular, 0-face $\{\mathbf{x}\}$ of $\mathcal{P}$ is called a vertex of $\mathcal{P}$. And the set of all vertices of $\mathcal{P}$ is said to be a vertex set of $\mathcal{P}$.

REMARK 2.4. Let $\mathcal{P}$ be a convex d-polytope of $\mathbb{R}^N$ and $V = \{\mathbf{x}_1, \cdots, \mathbf{x}_\nu\}$ a vertex set of $\mathcal{P}$. Since $\mathcal{P} = \mathrm{CONV}(V)$ and $dim\mathcal{P} = dimAFF(\mathcal{P})$, the maximal number of linearly independent elements which are contained in $\{\mathbf{x}_2 - \mathbf{x}_1, \cdots, \mathbf{x}_\nu - \mathbf{x}_1\}$ is d.

A polyhedral complex Γ of $\mathbb{R}^N$ is a finite set of polytopes such that

i) if $\mathcal{P} \in \Gamma$, then all faces of $\mathcal{P}$ are contained in Γ

ii) if $\mathcal{P}, \mathcal{Q} \in \Gamma$, then $\mathcal{P} \cap \mathcal{Q}$ is a face of $\mathcal{P}$ and also of $\mathcal{Q}$.

A polytope $\mathcal{P} \in \Gamma$ is said to be a face of Γ. In particular, if the dimension of $\mathcal{P}$ is i, then $\mathcal{P}$ is called an i-face of Γ. And a dimension of Γ is defined by

$$dim\Gamma = max\{dim\mathcal{P} | \quad \mathcal{P} \in \Gamma\}.$$

Let Γ, Γ' be polyhedral complexes. If all faces of Γ are contained in Γ' then Γ is called a subcomplex of Γ'. The set $\cup_{\mathcal{P}\in\Gamma}\mathcal{P}(\subset \mathbb{R}^N)$ is said to be the geometric realization of Γ which is denoted by $|\Gamma|$.

For $1 \leqq d \leqq N$, we define d-sphere $\mathbb{B}^d$ and $(d-1)$-spherical surface $\mathbb{S}^{d-1}$ by $\mathbb{B}^d$

$$= \{(x_1, x_2, \cdots, x_N) \in \mathbb{R}^N | x_1^2 + \cdots + x_d^2 \leq 1, x_{d+1} = \cdots = x_N = 0\}$$

and $\mathbb{S}^{d-1}$

$$= \{(x_1, x_2, \cdots, x_N) \in \mathbb{R}^N | x_1^2 + \cdots + x_d^2 = 1, x_{d+1} = \cdots = x_N = 0\}.$$

EXAMPLE 2.5. (1). Let $\mathcal{P}$ be a d-dimensional convex polytope of $\mathbb{R}^N$. $\mathcal{P}$ and all its faces form a polyhedral complex. If we denote it by Γ, then $|\Gamma| \simeq_{homeo} \mathbb{B}^d$.

(2). Let Γ be the set of all proper faces of a d-dimensional convex polytope $\mathcal{P}$. Then Γ is a polyhedral complex. Furthermore, we have $|\Gamma| \simeq_{homeo} \mathbb{S}^{d-1}$.

DEFINITION 2.6. (1). A d-dimensional convex polytope $\mathcal{P} \subset \mathbb{R}^N$ is said to be a d-simplex if $\mathcal{P}$ has exactly $d+1$ vertices.

(2). We say that a d-dimensional polyhedral complex Γ is pure if every maximal face of Γ has the same dimension.

(3). Simplicial complex of $\mathbb{R}^N$ is a polyhedral complex Δ such that all faces of Δ are simplex.

REMARK 2.7. (1). Let $\mathcal{P}$ be a d-simplex with vertex set $V = \{\mathbf{x}_0, \cdots, \mathbf{x}_d\}$. Then $\mathbf{x}_1 - \mathbf{x}_0, \mathbf{x}_2 - \mathbf{x}_0, \cdots, \mathbf{x}_d - \mathbf{x}_0$ must be linearly independent.

(2). If $\mathbf{x}_1 - \mathbf{x}_0, \mathbf{x}_2 - \mathbf{x}_0, \cdots, \mathbf{x}_d - \mathbf{x}_0$ are linearly independent, then CONV$(\{\mathbf{x}_0, \cdots, \mathbf{x}_d\})$ is a convex polytope of dimension d with vertices $\mathbf{x}_0, \cdots, \mathbf{x}_d$. Thus CONV$(\{\mathbf{x}_0, \cdots, \mathbf{x}_d\})$ is a d-simplex.

3. Main Results

From the elementary properties of d-simplex, we have the following result.

LEMMA 3.1. *Let $\mathcal{P}$ be a d-simplex with vertex set $V = \{\mathbf{x}_0, \cdots, \mathbf{x}_d\}$. Then $\mathcal{F}$ is a k-face of $\mathcal{P}$ if and only if $\mathcal{F} = CONV(\{\mathbf{x}_{i_0}, \cdots, \mathbf{x}_{i_k}\})$ for $\mathbf{x}_{i_j} \in V$, $j = 0, 1, \cdots, k$.*

Proof. (Necessity) Since the vertex set of $\mathcal{F}$ is $V \cap \mathcal{F}$, we may assume that $V \cap \mathcal{F} = \{\mathbf{x}_0, \cdots, \mathbf{x}_h\}$. But since $\mathbf{x}_1 - \mathbf{x}_0, \mathbf{x}_2 - \mathbf{x}_0, \cdots, \mathbf{x}_d - \mathbf{x}_0$ are linearly independent, $\mathbf{x}_1 - \mathbf{x}_0, \mathbf{x}_2 - \mathbf{x}_0, \cdots, \mathbf{x}_h - \mathbf{x}_0$ are also linearly independent. From $dim\mathcal{F} = k$, we deduce $h = k$. i.e. the vertex set of $\mathcal{F}$ is $\{\mathbf{x}_0, \cdots, \mathbf{x}_k\}$. Hence $\mathcal{F} =$ CONV$(\{\mathbf{x}_0, \cdots, \mathbf{x}_k\})$.

(Sufficiency) For $\mathbf{x}_{i_j} \in V$, $j = 0, 1, \cdots, k$, $\mathcal{F} =$ CONV$(\{\mathbf{x}_{i_0}, \cdots, \mathbf{x}_{i_k}\})$ is a face of $\mathcal{P}$. But from the fact that $\mathbf{x}_{i_1} - \mathbf{x}_{i_0}, \cdots, \mathbf{x}_{i_k} - \mathbf{x}_{i_0}$ are linearly independent, $\mathcal{F} =$ CONV$(\{\mathbf{x}_{i_0}, \cdots, \mathbf{x}_{i_k}\})$ is a k-face of $\mathcal{P}$. □

DEFINITION 3.2. Let Δ be a d-dimensional simplicial complex and σ a face of Δ. Then the link of σ in Δ, denoted $link_\Delta(\sigma)$, is the set of faces τ of Δ satisfying

i) $\sigma \cap \tau = \phi$

ii) there exists a face of Δ containing σ and τ.

Now, we are ready to prove the following main theorem.

THEOREM 3.3. *Let Δ be a simplicial complex consisting of a d-simplex $\mathcal{P}$ and its all faces. If σ is a k-face of Δ, then*

$$dim(link_{\Delta}(\sigma)) = d - (k+1).$$

Proof. Let $V = \{\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_d\}$ be the vertex set of Δ. From lemma 3.1, we may assume that

$$\sigma = CONV(\{\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_k\}), \quad k \leq d.$$

Let

$$W = V - \{\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_k\} := \{\mathbf{w}_0, \cdots, \mathbf{w}_{d-k-1}\}.$$

Fix $\mathbf{w} \in W$. By lemma 3.1, CONV($\{\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_k, \mathbf{w}\}$) is a $(k+1)$-face of Δ. So for each subset U of W,the convex closure of the set $\{\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_k\} \cup U$ is a face of Δ containing σ and CONV(U)(lemma 2.3). But since σ and CONV(U) are disjoint, CONV(U) is contained in $link_{\Delta}(\sigma)$. Applying lemma 3.1, each element of $link_{\Delta}(\sigma)$ is of the form CONV(U) for some subset U of W, and so we have

$$link_{\Delta}(\sigma) = \{CONV(U) | \quad U \subseteq W\}.$$

Since V is a vertex set of d-simplex $\mathcal{P}$, $\mathbf{x}_1 - \mathbf{x}_0, \cdots, \mathbf{x}_d - \mathbf{x}_0$ are linearly independent, and so $\mathbf{w}_1 - \mathbf{w}_0, \cdots, \mathbf{w}_{d-k-1} - \mathbf{w}_0$ are linearly independent. Hence CONV(W) is a simplex of dimension $d - (k+1)$. Therefore we have

$$dim(link_{\Delta}(\sigma)) = dim(CONV(W)) = d - (k+1).$$

□

Furthermore, theorem 3.3 can be extended to the case of pure simplicial complex, that is, COROLLARY 3.4. *Let Δ be a pure simplicial complex of dimension d. If σ is a k-face of Δ, then $dim(link_{\Delta}(\sigma)) = d - (k+1)$.*

Proof. Since Δ is a pure simplicial complex of dimension d, we may assume that Δ consists of finite d-simplexes $\mathcal{P}_1, \mathcal{P}_2, \cdots, \mathcal{P}_n$ (namely maximal faces) and their all proper faces. For each $1 \leq i \leq n$, let Δ_i be the simplical complex consisting of $\mathcal{P}_i$ and all its faces. Then $\Delta = \Delta_1 \cup \cdots \cup \Delta_n$. Let σ be a k-face of Δ. If we let $dim(link_{\Delta}(\sigma)) = h$, then there exists a h-face τ of Δ which is contained in $link_{\Delta}(\sigma)$. By definition of link, there exists a face $\mathcal{F}$ of Δ which contains σ and τ. Now, we may choose a subcomplex Δ_k of Δ which contains $\mathcal{F}$. This shows that $\tau \in link_{\Delta_k}(\sigma)$. Hence

$$(1) \qquad dim(link_{\Delta}(\sigma)) = h = dim\tau \leq dim(link_{\Delta_k}(\sigma)).$$

But since Δ is pure simplicial complex, $dim\Delta_k = dim\Delta = d$. Applying theorem 3.3, we have $dim(link_{\Delta_k}(\sigma)) = d - (k+1)$. So by (1), $h \leq d - (k+1)$. Clearly $d - (k+1) \leq dim(link_{\Delta}(\sigma))$. Hence $dim(link_{\Delta}(\sigma)) = d - (k+1)$. □

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