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ON THE INFLUENCE OF THE SECOND COEFFICIENT ON STARLIKE AND CONVEX FUNCTIONS OF COMPLEX ORDER

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Abstract

In this paper, we study how the second coefficient in the power series expansion of functions in $P_n(a;b,M)$, $F_n(a;b,M)$ and $G_n(a;b,M)$ influences certain properties like distortion, γ- spiral radius and radius of starlikeness of these classes.

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1. Introduction

Let $P_n(b,M)$, for $n \geq 1$, $n \in N = \{1,2,3,\cdots\}$ denote the class of functions of the form

$$(1.1) \qquad p(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \cdots, \quad c_n \neq 0$$

analytic in the unit disc $E = \{z \in \ker \mathbb{C} : |z| < 1\}$ and satisfying the condition

$$(1.2) \qquad \left|\frac{b-1+p(z)}{b} - M\right| < M, \quad z \in E$$

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where b is a non-zero complex number and $M > \frac{1}{2}$. Using Schwarz lemma, it is easily seen that (1.2) is equivalent to

$$(1.3) \qquad p(z) = \frac{1 + \{b(m+1) - m\} \cdot w(z)}{1 - m \cdot w(z)}, \quad m = 1 - \frac{1}{M}$$

where $w(z)$ is analytic in E and satisfies the conditions $w(0) = 0$, $|w(z)| \leq |z|$ for $z \in E$.

It can be easily seen that the image of E under a function in $P_n(b, M)$, $M \neq 1$ lies in a disc of radius $|b|/(1-m)$ and centre $\{1 + m(b-1)\}/(1-m)$. Thus function in $P_n(b, M)$ have positive real part in the unit disc E.

For non-zero complex number b and $M > \frac{1}{2}$, we now define the following subclasses of analytic functions in E as follows:

$F_n(b, M) =$

$$\{f(z) = z + a_{n+1}z^{n+1} + \cdots : zf'(z)/f(z) \in P_n(b, M), \ z \in E\}$$

and

$G_n(b, M) =$

$$\{f(z) = z + a_{n+1}z^{n+1} + \cdots : 1 + (zf''(z)/f'(z)) \in P_n(b, M), z \in E\}.$$

Clearly for $n = 1$, $F_n(b, M)$ reduces to the class $F(b, M)$ of bounded starlike functions of complex order, while $G(b, M)$ yields the class $G_n(b, M)$ of bounded convex functions of complex order. These classes were introduced and studied by Nasr and Aouf [12, 13].

If $p(z)$, defined by (1.1) belongs to the class $P_n(b, M)$ and $\theta = \exp(-i arg c_n)$, then $p(\theta z) = 1 + |c_n|z^n + \cdots \in P_n(b, M)$. Thus, we observe that there is no loss of generality by limiting our study to functions in $P_n(b, M)$ with non-negative real coefficients. It will be shown in section 2 that if $p(z)$ defined by (1.1) belongs to $P_n(b, M)$, then

$$|c_k| \leq |b| \cdot (m+1), \quad m = 1 - \frac{1}{M}$$

for each $k \geq n$.

With these observations, we define a subclass of $P_n(b, M)$, namely

$P_n(a; b, M) =$

$$\{p(z) = 1 + c_n z^n + \cdots \in P_n(b, M); \ \ |c_n| = a|b|(m+1), \ 0 \leq a \leq 1\}.$$

We now consider the following subclasses of analytic functions with fixed second coefficient generated from $P_n(a; b, M)$, viz.,

$F_n(a; b, M) = \{f(z) =$

$$z + \frac{a|b|(m+1)}{n} z^{n+1} + \cdots : \frac{zf'(z)}{f(z)} \in P_n(a; b, M) 0 \leq a \leq 1\}$$

and

$$G_n(a; b, M) = \{f(z) =$$

$$z + \frac{a|b|(m+1)}{n(n+1)} z^{n+1} + \cdots : 1 + \frac{zf''(z)}{f'(z)} \in P_n(a; b, M), 0 \leq a \leq 1\}.$$

It follows from the above definitions that $f(z) \in G_n(a; b, M)$, if and only if $zf'(z) \in F_n(a; b, M)$.

Following Silverman and Telage [15], we define another subclass of $P_n(a; b, M)$ as follows:

$$H_n(\mu; b, M) = \{f(z) =$$

$$z + \frac{\mu}{n} z^{n+1} + \cdots : \frac{zf'(z)}{f(z)} \in P_n(a; b, M) 0 \leq \mu \leq |b|(m+1)\}.$$

We note that for $a = \mu/|b|(m+1)$, the results for the classes $F_n(a; b, M)$ and $H_n(\mu; b, M)$ concide. For $n = 1$, we denote $P_n(b, M)$, $P_n(a; b, M)$, $F_n(a; b, M)$, $G_n(a; b, M)$ and $H_n(\mu; b, M)$ by $P(b, M)$, $P(a; b, M)$, $F(a; b, M)$, $G(a; b, M)$ and $H(\mu; b, M)$ respectively.

It is easily seen that by assigning specific values to the parameters n, a, b, and M, we obtain the following important subclasses of analytic functions studied by various authors in earlier work.

(i) $F_n(a; 1-\alpha, \frac{1}{2(1-\beta)}) \equiv S_n^\sharp(a; \alpha, \beta) (0 \leq \alpha < 1, \ 0 < \beta \leq 1)$, the class of studied by Kapoor and Mishra [7].

(ii) $F(a; (1-\beta)e^{-i\lambda}. \cos\lambda, M) \equiv F_a(\lambda; \beta, M)$ $(0 \leq \beta \leq 1, |\lambda| < \frac{\pi}{2})$, the class studied by Aouf [2].

(iii) $F(a; (1-\alpha)e^{-i\lambda}. \cos\lambda, \frac{1}{2\beta(1-\alpha)} \equiv S_a(\alpha; \beta, \lambda)$, $(0 \leq \alpha < 1, 0 \leq \beta < 1, |\lambda| < \pi/2)$, the class of λ-spirallike functions of order α and type β with fixed second coefficient studied by Ahuja [1].

In the present paper, we shall investigate how the second coefficient in the power series expansion of the functions in the classes $P_n(a; b, M)$ and $F_n(a; b, M)$ affects certain properties such as distortion, γ-spiral radius and radius of starlikeness of these classes. Similar results are also obtained for the class $G_n(a; b, M)$. We observe that for $a = 1$ our results generalize the work of Ahuja [1], Mogra and Ahuja [11] and for $a \neq 1$ the results are otherwise an improvement.

2. Growth Theorems

We first find the growth theorem for the class $P_n(a; b, M)$ which will be then used to establish the distortion theorems for the classes $F_n(a; b, M)$ and $G_n(a; b, M)$ respectively.

We need the following lemmas.

LEMMA 1. *Let* $p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \in p(b, M)$. *Then for* $k \geq 1$,

$$(2.1) \qquad |c_k| \leq |b| \cdot (m+1),$$

where $m = 1 - \frac{1}{M}$. *The estimate (2.1) is sharp.*

Proof. Since $p(z) \in P_n(b, M)$, we have

$$(2.2) \qquad p(z) = \frac{1 + \{b(m+1) - m\} \cdot w(z)}{1 - m \cdot w(z)},$$

where $w(z)$ is analytic in E and satisfies the condition $w(0) = 0$, $|w(z)| \leq 1$ for $z \in E$. On replacing the power series expansion of

$p(z)$ in (2.2) and simplifying the resulting equation, we get

$$\sum_{j=1}^{\infty} c_j z^j = \{b(m+1) + m \cdot \sum_{j=1}^{\infty} c_j z^j\} \cdot w(z). \tag{2.3}$$

Putting $r_k(z) = \sum_{j=1}^{k} c_j z^j$ and $R_{k-1}(z) = b(m+1) + m \cdot \sum_{j=1}^{k-1} c_j z^j$, the equation (2.3) can be written as

$$\begin{aligned} r_k(z) + \sum_{j=k+1}^{\infty} c_j z^j &= \{R_{k-1}(z) + \sum_{j=k}^{\infty} c_j z^j\} \cdot w(z) \\ &= R_{k-1}(z)w(z) + \sum_{j=k+1}^{\infty} \alpha_j z^j, \end{aligned}$$

for some suitable complex constants $\alpha_j's$. We note that the series $\sum_{j=k+1}^{\infty} c_j z^j$ and $\sum_{j=k+1}^{\infty}$ are absolutely and uniformly convergent in compacta on E. Now,

$$r_k(z) + \sum_{j=k+1}^{\infty} (c_j - \alpha_j) z^j = R_{k-1}(z) \cdot w(z).$$

Using the fact that $|w(z)| \leq 1$, squaring and then integrating both sides of the above equation, we obtain

$$\sum_{j=1}^{k} |c_j|^2 r^{2j} + \sum_{j=k+1}^{\infty} |c_j - \alpha_j|^2 r^{2j} \leq \{|b|(m+1)\}^2 + (\sum_{j=1}^{k-1} |c_j|^2 r^{2j}) \cdot m^2$$

Taking limit as $r \to 1$, we get

$$\sum_{j=1}^{k} |c_j|^2 \leq \sum_{j=1}^{k} |c_j|^2 + \sum_{j=k+1}^{\infty} |c_j - \alpha_j|^2 \leq \{|b|(m+1)\}^2 + m^2 \cdot \sum_{j=1}^{k-1} |c_j|^2$$

or,

$$|c_k|^2 \leq \{|b|(m+1)\}^2 - (1 - m^2) \sum_{j=1}^{k-1} |c_j|^2$$

Since $-1 < m \leq 1$, it follows that

$$|c_k| \leq |b|(m+1), \quad k \geq 1.$$

This completes the proof of the lemma. The estimate (2.1) is sharp for the functions

$$P_j(z) = \frac{1 + \{b(m+1) - m\}\varepsilon z^j}{1 - \varepsilon m z^j}, \quad |\varepsilon| = 1, \quad j \geq 1.$$

The following result is an immediate consequence of the above lemma.

COROLLARY 1. *If the function $p(z)$ defined by (1.1) belongs to the class $P_n(b, M)$, then*

$$|c_k| \leq |b|(m+1), \quad k \geq n.$$

The estimate is sharp.

We have the following iterated form of Schwarz lemma.

LEMMA 2. *[7]. Let $w(z) = z_n z^n + \cdots, \quad n > 1$ be analytic in E and maps E onto itself. Then for $|z| = r < 1$,*

$$|w(z)| \leq \frac{r^n(r + |a_n|)}{1 + |a_n| r}. \tag{2.4}$$

The estimate is sharp.

THEOREM 1. *Let $p(z) = 1 + a|b|(m+1)z^n + \cdots$ be in $P_n(a; b, M)$, $0 \leq a \leq 1$. Then for $|z| = r < 1$,*

$$M_2(r) \leq |p(z)| \leq M_1(r) \tag{2.5}$$

where

$$M_1(r) = 1 + \frac{(m+1)r^n(r+a)\{|b|(1+ar) + m \cdot Re(b) r^n(r+a)\}}{(1+ar)^2 - m^2 r^{2n}(r+a)^2} \tag{2.6}$$

and
(2.7)
$$M_2(r) = 1 - \frac{(m+1)r^n(r+a)\{|b|(1+ar) - m \cdot Re(b)r^n(r+a)\}}{(1+ar)^2 - m^2r^{2n}(r+a)^2}.$$

The estimates in (2.5) are sharp. Further, if

$$Q_n(r,a) = \max_{p(z)\in P_n(a;b,M)} \{\max_{|z|=r} |p(z)|\}$$

and

$$q_n(r,a) = \max_{p(z)\in P_n(a;b,M)} \{\max_{|z|=r} |p(z)|\},$$

then $Q_n(r,a)$ *is increasing in* a, $q_n(r,a)$ *is decreasing in* a, $Q_n(r,0) = Q_{n+1}(r,1)$ *and* $q_n(r,0) = q_{n+1}(r,1)$.

Proof. Since $p(z) \in P_n(a;b,M)$, we have

$$p(z) = \frac{1 + \{b(m+1) - m\} \cdot w(z)}{1 - m \cdot w(z)}, \quad m = 1 - \frac{1}{M}$$

for which it follows that

$$w(z) = \frac{1-p(z)}{m(1-p(z)) - b(m+1)}$$
(2.8)
$$= -a\exp(i \cdot arg(b))z^n + \cdots.$$

We observe that $w(z)$ is an analytic map of E onto itself and $|-a \cdot \exp(i \cdot arg(b))| = a \leq 1$. Then by Lemma 2, it follows that

(2.9)
$$|w(z)| \leq \frac{r^n(r+a)}{1+ar}, \quad |z| = r < 1.$$

Thus, by using (2.9) in (2.8), we have

$$\left|\frac{1-p(z)}{m(1-p(z)) - b(m+1)}\right| \leq \frac{r^n(r+a)}{1+ar}.$$

The above inequality implies that $p(z)$ maps the disc $|z| \leq r$ onto a disc $|p(z) - C| \leq R$, where

$$C = 1 + \frac{m(m+1)b \cdot r^{2n}(r+a)^2}{(1+ar)^2 - m^2 \cdot r^{2n}(r+a)^2}$$

and

$$R = \frac{(m+1)|b|r^{2n}(r+a)(1+ar)}{(1+ar)^2 - m^2 \cdot r^{2n}(r+a)^2}.$$

From this it immediately follows that

$$M_2(r) \leq Re\{p(z)\} \leq |p(z)| \leq M_1(r),$$

where $M_1(r)$ and $M_2(r)$ are defined as in the theorem.

The right hand inequality in (2.5) is sharp for the function

$$p_1(z) = \frac{1 + a \cdot z + aAz^n + A \cdot z^{n+1}}{1 + az + amz^n + m \cdot z^{n+1}} \tag{2.10}$$

at the point $z = r$; while the left hand side inequality in (2.5) is attained for the function

$$p_2(z) = \frac{1 + a(\bar{\varepsilon} + Az^{n-1})z + \bar{\varepsilon}Az^{n+1}}{1 + a(\bar{\varepsilon} + mz^{n-1})z + \bar{\varepsilon}mz^{n+1}} \tag{2.11}$$

at the point $z = \varepsilon r$, where $A = m - b(m+1)$ and $\varepsilon = \exp(i\pi/n)$.

This proves the first part of the theorem.

To prove the remaining part of the theorem, we observe that $Q_n(r,a) = M_1(r)$ and $q_n(r,a) = M_2(r)$. Thus,

$$\frac{\partial}{\partial a}\{Q_n(r,a)\} \geq \frac{(m+1)|b|r^n(1-r^2)}{(1+ar) + mr^n(r+a)} > 0$$

and

$$\frac{\partial}{\partial a}\{q_n(r,a)\} \leq -\frac{(m+1)|b|r^n(1-r^2)}{(1+ar) + m \cdot r^n(r+a)} < 0.$$

Hence $Q_n(r,a)$ is increasing in a and $q_n(r,a)$ is decreasing in a. Finally, $Q_n(r,0) = Q_{n+1}(r,1)$ and $q_n(r,0) = q_{n+1}(r,1)$ follows from the definitions of these functions. This proves the theorem.

THEOREM 2. *Suppose* $f(z) = z + \frac{a|b|\cdot(m+1)}{n} z^{n+1} + \cdots$ *belongs to* $F_n(a; b, M)$. *Then for* $|z| = r < 1$,

$$|f(z)| \le r \cdot \exp\{(m+1)\int_0^r \Phi_1(t)dt\} \tag{2.12}$$

and

$$|f(z)| \le r \cdot \exp\{-(m+1)\int_0^r \Phi_2(t)dt\}, \tag{2.13}$$

where

$$\Phi_1(t) = \frac{t^{n-1}(t+a)\{|b|(1+at) + m \cdot Re(b)t^n(t+a)\}}{(1+at)^2 - m^2 \cdot t^{2n}(t+a)^2} \tag{2.14}$$

and

$$\Phi_2(t) = \frac{t^{n-1}(t+a)\{|b|(1+at) - m \cdot Re(b) \cdot t^n(t+a)\}}{(1+at)^2 - m^2 \cdot t^{2n}(t+a)^2}. \tag{2.15}$$

The estimates in (2.12) and (2.13) are sharp.

Proof. Since Theorem 1 remains valid with $p(z)$ replaced by $Re\{p(z)\}$ in (2.5), using the definition of the class $F_n(a; b, M)$, we have for $|z| = r < 1$,

(2.16)

$$Re\{\frac{zf'(z)}{f(z)} \le 1 + \frac{(m+1)r^n(r+a)\{|b|(1+ar) + mRe(b)r^n(r+a)\}}{(1+ar)^2 - m^2 \cdot r^{2n}(r+a)^2}$$

and

(2.17)

$$Re\{\frac{zf'(z)}{f(z)} \ge 1 - \frac{(m+1)r^n(r+a)\{|b|(1+ar) - mRe(b)r^n(r+a)\}}{(1+ar)^2 - m^2 \cdot r^{2n}(r+a)^2}.$$

Letting $z = re^{i\theta}(0 \le \theta \le 2\Pi)$, we note that

$$\begin{aligned}\log(|\frac{f(z)}{z}|) &= Re\int_0^z \{\frac{f'(s)}{f(s)} - \frac{1}{s}\}ds \\ &= \int_0^r \frac{1}{t} \cdot Re\{te^{i\theta}\frac{f'(te^{i\theta})}{f(te^{i\theta})} - 1\}dt\end{aligned}$$

and using (2.16), we deduce that

$$\log\left|\frac{f(z)}{z}\right| \leq$$

$$(m+1)\cdot\int_0^r \frac{t^{n-1}(t+a)\cdot\{|b|(1+at)+m\cdot Re(b)t^n(t+a)\}}{(1+at)^2-m^2\cdot t^{2n}(t+a)^2}dt$$

which implies (2.12). Again, in view of

$$\log\left|\frac{f(z)}{z}\right| = Re\{\log(\frac{f(z)}{z})\} = \int_0^r Re\{\frac{\partial}{\partial t}\log(\frac{f(t)}{t})\}dt$$
$$= \int_0^r \frac{1}{t}Re\{\frac{tf'(t)}{t}-1\}dt$$

and with the aid of (2.17), we may write

$$\log\left|\frac{f(z)}{z}\right| \geq$$

$$-(m+1)\cdot\int_0^r \frac{t^{n-1}(t+a)\cdot\{|b|(1+at)-m\cdot Re(b)t^n(t+a)\}}{(1+at)^2-m^2\cdot t^{2n}(t+a)^2}dt$$

which gives (2.13).

Equality in (2.12) holds for the function $f_1(z)$ with $f_1(0)=0$ and

$$f_1(z) = z\cdot\exp\{(m+1)\cdot\int_0^z \Phi_1(t)dt\}$$

at the point $z=r$. For equality in (2.13), consider the function $f_2(z)$ with $f_2(0)=0$ and

$$f_2(z) = z\cdot\exp[(m+1)$$

$$\cdot\int_0^z \frac{t^{n-1}(\bar{\varepsilon}t+a)\{|b|(1+\bar{\varepsilon}at)+m\cdot Re(b)t^n(\bar{\varepsilon}t+a)\}}{(1+\bar{\varepsilon}at)^2-m^2\cdot t^{2n}(\bar{\varepsilon}t+a)^2}dt],$$

where $\varepsilon=\exp(i\Pi/n)$. At the point $z=\varepsilon r$,

$$\left|\frac{f_2(z)}{z}\right| = \exp[(m+1)$$

$$\cdot \int_0^r \frac{t^{n-1}(\bar{\varepsilon}t+a)\{|b|(1+\bar{\varepsilon}at)+m\cdot Re(b)t^n(\varepsilon t+a)\}}{(1+\varepsilon at)^2-m^2\cdot t^{2n}(\varepsilon t+a)^2}dt].$$

Since the integration is independent of the path, we consider the integration along the ray joining 0 to εr. Thus,

$$|f_2(\varepsilon r) = r\cdot \exp[-(m+1)$$

$$\cdot \int_0^r \frac{t^{n-1}(t+a)\{|b|(1+at)+m\cdot Re(b)t^n(t+a)\}}{(1+at)^2-m^2\cdot t^{2n}(t+a)^2}dt].$$

Thus, equality holds in (2.13) for the function $f_2(z)$. That this function is in $F_n(a;b,M)$ can be seen by considering $Re\{z\frac{d}{dz}f(z)\}$. This completes the proof of Theorem 2.

COROLLARY 2. *Suppose* $f(z)=z+\frac{a|b|(m+1)}{n}z^{n+1}+\cdots$ *belongs to* $F_n(a;b,M)$. *Then for* $|z|=r<1$,

$$|f'(z)|\le M_1(r)\cdot \exp\{(m+1)\cdot \int_0^r \Phi_1(t)dt\}$$

and

$$|f'(z)|\ge M_2(r)\cdot \exp\{-(m+1)\cdot \int_0^r \Phi_2(t)dt\},$$

where $M_1(r)$, $M_2(r)$ *are defined as in Theorem 1 and* $\Phi_1(t)$, $\Phi_2(t)$ *are defined as in Theorem 2.*

The estimates are sharp.

Proof. The corollary follows immediately from Theorem 1, Theorem 2 and in view of the fact tat $f(z)\in F_n(a;b,M)$, if and only if $zf'(z)/f(z)\in P_n(a;b,M)$.

COROLLARY 3. *Let* $f(z)=z+\frac{ab(m+1)}{n}z^{n+1}+\cdots$ *be in the class* $F_n(a,b,M)$, $b>0$. *Then for* $|z|=r<1$,

$|f(z)|\le r\cdot \exp\{(m+1)b\cdot \int_0^r \frac{t^{n-1}(t+a)}{(1+at)-m\cdot t^n(t+a)}dt\}$,

$|f(z)|\ge r\cdot \exp\{-(m+1)b\cdot \int_0^r \frac{t^{n-1}(t+a)}{(1+at)+m\cdot t^n(t+a)}dt\}$,

$|f'(z)|\le M_3(r)\cdot \exp\{(m+1)b\cdot \int_0^r \frac{t^{n-1}(t+a)}{(1+at)-m\cdot t^n(t+a)}dt\}$,

and

$|f'(z)| \leq M_4(r) \cdot \exp\{-(m+1)b \cdot \int_0^r \frac{t^{n-1}(t+a)}{(1+at)+m \cdot t^n(t+a)} dt\}$,
where

$$M_3(r) = 1 + \frac{(m+1)b \cdot r^n(r+a)}{(1+ar) - m \cdot r^n(r+a)}$$

and

$$M_4(r) = 1 - \frac{(m+1)b \cdot r^n(r+a)}{(1+ar) + m \cdot r^n(r+a)}.$$

All the estimates are sharp.

COROLLARY 4. *Let $f(z) = z + a(1-\alpha)z^3 + \cdots$ be an odd starlike function of order $\alpha (0 \leq \alpha < 1)$. Then for $|z| = r < 1$,*

$$|f(z)| \leq r \cdot [(1-r)^{(1+a)} \cdot \{1 + (1+a)r + r^2\}^{-\frac{2(1-\alpha)}{3+a}},$$

and

$$|f(z)| \geq r \cdot [(1-r)^{(1-a)} \cdot \{1 - (1-a)r + r^2\}^{-\frac{2(1-\alpha)}{3+a}}.$$

The estimates are sharp.

Proof. Putting $n = 2$, $m = 1$ and $b = (1-\alpha)$, $0 \leq \alpha < 1$ in Corollary 3, the results follws after some simplifications.

COROLLARY 5. *Let $f(z) = z + \frac{a|b|(m+1)}{n} z^{n+1} + \cdots$ be in $0 \leq a \leq 1$ $F_n(a; b, M)$. Then for $|z| = r < 1$ and $m \neq 0$,*

$$|f(z)| \leq r \cdot [\frac{(1+mr^n)^{\frac{b-Re(b)}{n}}}{(1-mr^n)^{\frac{b+Re(b)}{n}}}]^{(m+1)/2n} \tag{2.18}$$

and

$$|f(z)| \geq r \cdot [\frac{(1-mr^n)^{\frac{b-Re(b)}{n}}}{(1+mr^n)^{\frac{b+Re(b)}{n}}}]^{(m+1)/2n} \tag{2.19}$$

where for $m = 0$,

$$|f(z)| \leq r \cdot \exp(\frac{|b| \cdot r^n}{n}) \tag{2.20}$$

and

$$|f(z)| \geq r \cdot \exp(-\frac{|b| \cdot r^n}{n}) \tag{2.21}$$

All the estimates are sharp.

Proof. Let the right hand side of (2.12) and (2.13) be denoted by $A_1 \equiv A_1(r,n,a)$ and $B_1 \equiv B_1(r,n,a)$ respectively. By Theorem 1, $A_1(r,n-1,0) = A_1(r,n,1)$ and $B_1(r,n-1,0) = B_1(r,n,1)$. Now, taking $a = 1$ in (2.12) and (2.13), we deduce (2.18), (2.19), (2.20) and (2.21) respectively.

To see that the inequalities are best possible, consider the function

$$f(z) = \begin{cases} \frac{z}{(1-mz^n)^{b(m+1)/nm}}, & m \neq 0 \\ z \cdot \exp(\frac{bz^n}{n}), & m = 0. \end{cases} \tag{2.22}$$

Equality in (2.18) and (2.20) holds for the function $f(z)$ at the point $z = r[\{(\bar{b}/b)^{1/2} + mr^n\} / \{1 + (\bar{b}/b)^{1/2} mr^n\}]^{1/n}$; while equality in (2.19) and (2.21) is attained by the same function $f(z)$ at the point $z = r[\{mr^n - (\bar{b}/b)^{1/2}\} / \{1 - (\bar{b}/b)^{1/2} mr^n\}]^{1/n}$.

REMARKS 1. *Setting* $b = (1-\alpha)$ *and* $M = 1/2(1-\beta)$ $(0 \leq \alpha < 1, 0 < \beta \leq 1)$ *in Corollary 2.3, we get the corresponding result obtained by Kapoor and Mishra [7] for the class* $S_n^\sharp(a;\alpha,\beta)$.

2. *Taking* $n = 1$, $b = (1-\alpha)\exp(-i\lambda)\cos\lambda$ *and* $M = 1/2(1-\beta)$ $(0 \leq \alpha < 1, 0 < \beta \leq 1, |\lambda| < \Pi/2)$ *in Corollary 2.4, we get the distortion theorem due to Mogra and Ahuja [11] which in turn leads to the result of Juneja and Mogra [6] for* $\lambda = 0$.

3. *Putting* $n = 1$ *and* $b = \exp(-i\lambda) \cdot \cos\lambda$ $(|\lambda| < \Pi/2)$ *in Corollary 2.3, we obtain the corresponding distortion theorem due to Goel [5].*

We can determine the bounds for $|f'(z)|$ for $f \in G_n(a;b,M)$ from those found for the class $F_n(a;b,M)$ by using the well known result $f \in G_n(a;b,M)$, if and only if $zf' \in F_n(a;b,M)$.

THEOREM 3. *Let* $f(z) = z + \frac{a|b|(m+1)}{n(n+1)} z^{n+1} + \cdots$ *be in* $G_n(a; b, M)$. *Then for* $|z| = r < 1$,

$$|f'(z)| \leq \exp\{(m+1)\int_0^r \Phi_1(t)dt\}$$

and

$$|f'(z)| \geq \exp\{-(m+1)\int_0^r \Phi_2(t)dt\},$$

where $\Phi_1(t)$ *and* $\Phi_2(t)$ *are defined as in theorem 2. The estimates are sharp.*

3. Radius of γ-spirallikeness

Let S denote the family of all normalized functions which are analytic and univalent in the unit disc E. Following Silverman and Telage [15], if $f \in S$ and $|\gamma| < \Pi/2$ then the γ-spiral radius of order $\delta (0 \leq \delta \leq 1)$ of $f(z)$, denoted $R(\gamma, \delta, f(z))$, is given by

$$R(\gamma, \delta, f(z)) = \sup\{r : Re(\exp^{i\gamma} \frac{zf'(z)}{f(z)} > \delta, \quad |z| = r\}.$$

If U is a subclass of S, then the γ-spiral radius of U, denoted by $R(\gamma, \delta, U)$ is given by

$$R(\gamma, \delta, U) = \inf_{f \in U}\{R(\gamma, \delta, f(z))\}. \tag{3.1}$$

These definitions reduce to those of Libera [9] when $\delta = 0$. If, in addition, $\gamma = 0$ then the right hand side of (3.1) is the radius of starlikeness of the family U.

We now determine the γ-spiral radius of order δ of the class $F_n(a; b, M)$.

THEOREM 4. *The* γ*-spiral radius of order* δ *of the class* $F_n(a; b, M)$, $M > \frac{1}{2}$ *is the smallest positive root of the equation*
$m\{(m+1)|b|\cos(\gamma+\theta) - m \cdot \cos(\gamma - \delta)\}r^{2n}(r+a)^2 -$
$-(m+1)|b| \cdot r^n(r+a)(1+ar) + (\cos\gamma - \delta)(1+ar)^2 = 0,$

where $\theta = arg(b)$. The result is sharp.

Proof. Suppose $f \in F_n(a; b, M)$. Then from the proof of Theorem 1

$$Re\{e^{i\gamma}\frac{zf'(z)}{f(z)}\} \geq Re(e^{i\gamma} \cdot C) - R, \tag{3.2}$$

where

$$C = 1 + \frac{m(m+1)b \cdot r^{2n}(r+a)^2}{(1+ar)^2 - m^2 \cdot r^{2n}(r+a)^2}$$

and

$$R = \frac{(m+1)|b| \cdot r^{2n}(r+a)(1+ar)}{(1+ar)^2 - m^2 \cdot r^{2n}(r+a)^2}.$$

Now, $f(z)$ is γ-spirallike of order δ, if the right hand side of (3.2)$\geq$ δ. This is equivalent to

$$(\cos\gamma - \delta)(1+ar)^2 - (\cos\gamma - \delta)m^2r^{2n}(r+a)^2 + m(m+1)|b| \cdot$$
$$\cos(\gamma+\theta)r^{2n}(r+a)^2 - (m+1)|b| \cdot r^n(r+a)(1+ar) \geq 0,$$

which on simplification and with the aid of (3.1) gives the required result

The result is sharp, the extremal function being given $f_2(z)$ of Theorem 2.

COROLLARY 5. *If $f \in F_n(b, M)$, $M > \frac{1}{2}$, then $f(z)$ is γ-spirallike for $|z| < r_0$, where r_0 is the smallest positive root of the equation*

$$m\{(m+1)|b|\cos(\gamma+\theta) - m \cdot \cos\gamma\}r^{2n} - (m+1)|b| \cdot r^n + \cos\gamma = 0,$$

where $\theta = arg(b)$. The result is sharp.

Proof. The result follows by setting $\delta = a - 1 = 0$ in Theorem 4. The estimate is sharp for the function f given by (2.22) at the point

$$z = -r[\{(\bar{b}/b)^{1/2}e^{-i\gamma} - mr^n\}/\{1 - (\bar{b}/b)^{1/2}e^{-i\gamma} \cdot mr^n\}]^{1/n}.$$

Taking $b = 1$ and $m = 1$ in Corollary 5, we have the following.

COROLLARY 6. *Suppose f is a starlike function in the unit disc E. Then $f(z)$ is γ-spirallike for $|z| < r(n,\gamma)$, where*

$$r(n,\gamma) = \{sec\gamma + |\tan\gamma|\}^{-\frac{1}{n}}.$$

The result is sharp for the function f given by

$$f(z) = \frac{z}{(1-z^n)^{2/n}}.$$

COROLLARY 7. *The γ-spiral radius of order δ of the class $H(\mu, b, M)$ is the smallest positive root of the equation*

$$(\cos\gamma-\delta)(u^2+m^2v^2\cdot r^{2n})-|b|(m+1)vr^n(u-m\cdot\cos(\gamma+\theta)\cdot vr^n) = 0,$$

where

$$u = \mu r + (m+1)|b| \text{ and } \quad v = \mu + (m+1)|b|r.$$

The result is sharp.

Proof. Letting $a = \mu/(m+1)|b|$ in Theorem 4, and writing $u = \mu r + (m+1)|b|$, $v = \mu + (m+1)|b|r$ in the resulting equation, we obtain the required result.

REMARKS 1. *Putting $n = 1$ and*

$$(b, M) = ((1-\alpha)e^{-i\lambda}\cdot\cos\lambda,\ \frac{1}{2(1-\beta)}),$$

$$(0 \le \alpha < 1,\ 0 < \beta \le 1,\ |\lambda| < \frac{\Pi}{2})$$

$$(b, M) = ((1-\alpha)e^{-i\lambda}\cdot\cos\lambda, M),\quad M > \frac{1}{2}$$

$$(b, M) = (e^{-i\lambda}\cos\lambda, \frac{1}{\cos\lambda})(|\lambda| < \Pi/2)$$

in Corollary 5, we get the corresponding results obtained by Mogra and Ahuja [11].

2. *Letting $n = 1$, $b = (1-\alpha)e^{-i\lambda}\cdot\cos\lambda$ and $M = \frac{1}{2(1-\beta)}$, $(0 \le \alpha < 1, 0 < \beta \le 1, |\lambda| < \Pi/2)$ in Theoren 4 and Corollary 5, we obtain the results due to Ahuka [1].*

By fixing $\gamma = 0$ in Theorem 4, we immediately obtain the following result.

THEOREM 5. *The radius of starlikeness of order δ of the class $F_n(a; b, M)$ is the smallest positive root of the equation*

$$m\{(m+1)Re(b) - m(1-\delta)\}r^{2n}(r+a)^2 - (m+1)|b|r^n(r+a)$$
$$(1+ar) + (1-\delta)(1+ar)^2 = 0.$$

The result is sharp.

Setting $a = 1$ and $\delta = 0$ in Theorem 5, we get the following result which in turn yields the result of Ahuja [1] for

$$b = (1-\alpha)e^{-i\lambda} \cdot \cos\lambda \quad \text{and} \quad M = \frac{1}{2(1-\beta)}$$

$$(0 \le \alpha < 1, 0 < \beta \le 1, |\lambda| < \frac{\Pi}{2}).$$

COROLLARY 8. *If $f \in F_n(b, M)$, then $f(z)$ is starlike in $|z| < r(n, b, M)$, where $r(n, b, M) =$*

$$2^{1/n}[(m+1)|b| + \sqrt{(m+1)|b| - 4m \cdot \{(m+1)Re(b) - m\}}]^{-1/n}$$

The result is sharp for the function f given by (2.22). Taking $b = e^{-i\lambda} \cdot \cos\lambda(|\lambda| < \Pi/2)$ and $M = \infty$ in Corollary 8, we deduce the following result which has been obtained independently using a different method by Robertson [14], Libera [9] and Libera and Ziegler [10] for $n = 1$.

COROLLARY 9. *Let $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$ be a λ-spiralike function. Then $f(z)$ is starlike for $|z| < r(n, \lambda)$, where*

$$r(n, \lambda) = \{\cos\lambda + |\sin\lambda|\}^{-1/n}$$

The estimate is sharp for the function f given by

$$F(z) = \frac{z}{(1-z^n)^{2 \cdot \exp(-i\lambda)/n}}.$$

REMARK. *1. For* $n = 1$ *and* $b = \exp(-i\lambda)\cos\lambda$ $(|\lambda| < \Pi/2)$, *Corollary 9 gives the radius of starlikeness for the class* $S^{\lambda}(M)$ *obtained by Ahuja [1].*

2. The preceeding results can also be obtained for the class $G_n(a; b, M)$ *from those found for the class* $F_n(a; b, M)$ *by a simple application of the fact that* $f \in G_n(a; b, M)$, *if and only if* $zf' \in F_n(a; b, M)$ *and*

$$Re\{e^{i\gamma}(1 + \frac{zf'(z)}{f(z)})\} > \delta \quad \text{iff} \quad Re\{e^{i\gamma}\frac{zg'(z)}{g(z)}\} > \delta,$$

where $g(z) = zf'(z)$, $z \in E$.

Further, by specializing the parameters a, b, n *and* M *in such results for the class* $G_n(a; b, M)$, *we can deduce several interesting special cases which concide with the work of Chichra [4], Kulshrestha [8], Libera and Ziegler [10] and many others.*

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