財務管理論叢 第3卷 第2號 (1996.10)

EVALUATION OF ILLIQUID ASSETS

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ABSTRACT

This paper studies evaluation of illiquid assets by using a consumption and investment model. In particular, the paper defines the marginal values of illiquid assets and shows how optimal consumption and investment policies are related to the marginal values. The general results are illustrated by two concrete examples.

1. Introduction

This paper studies evaluation of illiquid assets. An illiquid asset is an asset whose liquidation is costly or impossible. For example, there is no market for claims to individuals' future labor income, and therefore, human capital is an essentially

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** This paper is prepared for a presentation at the 1995 NBER Summer Institute Workshop on Asset Pricing. I would like to express my deep thanks John Campbell, Avinash Dixit, and Giuseppe Bertola for their guidance and suggestions. I also would like to thank Kerry Back, Angus Deaton, Phil Dybvig, Miles Kimball, Chris Lamoureux, Guy Laroque, Lars Svensson, Harald Uhlig and Ingrid Werner for valuable comments on the original draft. I also have benefited greatly from discussions with Darrell Duffie and Thaleia Zariphopoulou. Of course, all the remaining errors are mine.

illiquid asset. There are also highly illiquid financial securities. The bid-ask spread of small company stocks trading in the NASDAQ market is well above 10% and can be as high as 35% of their median prices (Lamoureux and Sanger (1989)). Many Japanese government bonds are also highly illiquid (Singleton (1994)).

One important problem facing an economic agent who owns an illiquid asset is a measurement problem—that of how to measure the value of the illiquid asset. The difficulty of the problem comes from that there is no single market price for an illiquid asset. For an illiquid financial asset there are two significantly different prices, i.e., the bid price and ask price. In this case there is no general principle that tells the agent which price must be chosen to measure the value of the illiquid asset. For a non-traded asset (e.g., human capital) the situation is even worse, because there is no market and no quoted price for the asset.

This paper investigates the evaluation problem for the case where the economic agent's preference is characterized by a homothetic utility function. Davis and Norman (1990) have studied a consumption and portfolio selection problem in the presence of transactions costs by studying the ratio of the marginal utility of human capital to the marginal utility of money. Koo (1994a,b) has studied the evaluation of human capital by examining the ratio of the marginal utility of human capital to the marginal utility of money. In this paper I extend these methods to study more general evaluation of illiquid assets. In particular, I show that in the presence of an illiquid asset, the economic agent's consumption and investment decisions are related to a measure of total wealth which is the sum of liquid assets and a measure of the value of the illiquid asset-the measurement is done by using the above-mentioned ratio of the marginal utility of human capital to the marginal utility of money.

The general results of this paper are illustrated by two examples. First, I consider the consumption and investment problem of an agent who owns shrares of a small company stock whose bid-ask spread is very large. This example estends the model investigated by Constantinides (1986) and Davis and Norman (1990) to include a portfolio selection problem between liquid risky assets. Second, I consider the consumption and investment problem of an agent who is endowed with an asset which is either non-traded (e.g., human capital) or can be liquidated only in its entirety (e.g., small propietorship). This example extends the consumption and portfolio selection problem with labor income (He and Pagés (1993), Duffie, Fleming and Zariphopoulou (1993), Cuoco (1994) Koo (1995a,b)).

As a byproduct product of this investigation, I provide a unified treatment to the previous research on non-traded assets (Deaton (1991), Svensson and Werner (1993), Duffie, Fleming and Zariphopoulu(1993), Koo (1994a,b) and research on transactions costs (Constantinides (1986), Davis and Norman(1990), Shreve and Soner (1992)). Namely, I show that optimal consumption and portfolio selection polices take the same form regardless of whether the illiquid asset is non-traded or subject to proportional transactions costs.

Karatzas, Lehoczky, Sethi, and Shreve (1986), Cox and Huang (1989), and He and Pearson (1991) have developed a martingale approach to consumption and investment problems. He and Pagés (1993) and Cuoco (1994) have used the martingale approach to solve the consumption and investment problem with labor income. My approach is different from that in these papers in the sence that I do not rely on the dual characterization of the consumption and investment problem whereas they do.

The paper proceeds as follows. Section 2 presents a consumption and investment problem and discusses the problem of evaluating illiquid assets. Section 3 studies two examples and section 4 concludes. Appendix gives the technical details.

2. A Consumption and Portfolio Selection Problem and Evaluation of Illiquid Assets

I start with a simple cosumption and portfolio selection problem faced by an infinitely lived agent who currently owns M illiquid assets $I_1, I_2, \ldots, I_{M.}$

The agent's objective is to maximize utility derived from consumption, which is given by the following von Neumann-Morgenstern time separable utility function:

$$U_t = E_t \int_t^{\infty} e^{-\delta_t} v(C(s))ds \tag{1}$$

where E_t is the expectation taken at time t and v is a constant relative risk aversion fuction

$$v(C) = \begin{cases} \dfrac{C^{1-\gamma}}{1-\gamma} & \text{if } \gamma \neq 1 \\ \log C & \text{if } \gamma = 1. \end{cases} \tag{2}$$

The illiquid assets considered in this paper can be classified into one of the following categories: (i) non-traded assets, e.g., human capital, (ii) assets that can be liquidated only in entireties, e.g., small private businesses, and (iii) financial assets that have large bid-ask spreads. I assume that $I_1, \ldots, I_{m_1}$ belong to the first category, $I_{m_1+1}, \ldots, I_m$ belong to the second category, and $I_{m+1}, \ldots, I_{m+1}$ belong to the third category.

The assets $I_1, \ldots, I_m$ generate cash flows equal to $Q_1(t)dt, \ldots, Q_m(t)$ during the infinitesimal time period $(t, t+dt)$. The liquidation value of an illiquid asset I_i for $i=m_1+1, \dots, m$ is assumed to be $A_i Q_i(t)$, where A_i is a constant. The face values of illiquid financial assets $I_{m+1}, \dots, I_M$ are defined to be the value for which one can buy these assets. The illiquid financial assets have face values $Q_{m+1}(t), \dots, Q_M(t)$ at time t and pay dividends $\nu_{m+1}Q_{m+1}(t), \dots, \nu_M Q_M(t)$ over the infinitesimal time period $(t, t+dt)$. When the agent sells an illiquid financial asset $I_i(m+1 \le i \le M)$, he or she gets liquid assets worth $(1-\lambda_i)Q_i(t)$, where λ_i is a constant.

$Q_1(t), \dots, Q_M(t)$ evolve according to the following stochatic differential equations

$$\frac{dQ_i(t)}{Q_i(t)} = \nu_{1,i}dt + \tau_i dB_i(t), \qquad i=1, \dots, M \tag{3}$$

where $B_1, B_2, \dots, B_M$ are Brownian motions with constant correlations.

There are $N+1$ liquid assets which the agent can buy and sell at their face values without incurring any costs of trading, among which one is riskless and the others are risky. I assume that the risk-free rate is a constant γ, i.e., the price $P_0(t)$ of the riskless asset follows a process

$$\frac{dP_0(t)}{P_0(t)} = \gamma dt. \tag{4}$$

The price $P_j(t)$ of the j-th liquid risky asset follows a geometric Brownian motion

$$\frac{dP_j(t)}{P_j(t)} = \mu_j dt + \sum_{k=1}^{N} \sigma_{jk} dZ_k(t), \qquad k=1,\ldots,N \tag{5}$$

where μ_j and σ_{jk}, for $j,k=1,\cdot\cdot\cdot,N$, are constants, and $Z_k(t)$ for $j=1,\cdot\cdot\cdot,N$ are independent Brownian motions jointly normal with $B_1(t), B_2(t),\cdot\cdot\cdot,B_N(t)$. I assume that all the correlations between these Brownian motions are constant. I also assume that the market for liquid assets is complete, i.e., the matrix $\Sigma=(\sigma_{ij})$ is non-singular. The assumption of a complete liquid asset market enables me to focus on market in completness which arises only in the form of illiquidity.

The bank accounts are liquid assets. Shares of large well-recognized stocks and mutual funds can also be regarded as almost liquid, because the commissions and information costs for these shares are currently very small and the bid-ask spreads are relatively small.

In the real world there is a limit to the amount by which an asset can be shorted. In this paper I consider short selling constraints of the following form

$$-\overline{\omega}_{1,j} \leq \omega_j(t) \leq \overline{\omega}_{2,j} \tag{6}$$

for some constants $0 < \overline{\omega}_{1,j}, \overline{\omega}_{2,j} \leq \infty$, $\quad j=1,2,\cdot\cdot\cdot,N.$

The agent's consumption can be financed only by liquid assets. Then, if the investor does not buy or sell any of the illiquid assets, liquid wealth $L(t)$, which is the total value of liquid assets held by the investor, evlves according to

$$\begin{aligned} dL(t) &= (\gamma L(t) - C(t) + \sum_{i=1}^{m} Q_i(t) + \sum_{i=m+1}^{M} \nu_i Q_i(t))dt \\ &\quad + \sum_{j=1}^{N} \omega_j(t)(\mu_j - \gamma)L(t)dt + \sum_{j,k=1}^{N} \omega_{j,t}\sigma_{j,k}L(t)dZ_k(t) \end{aligned} \tag{7}$$

where $C(t)$ is the investor's rate of consumption and $\omega_j(t)$ is the proportion of liquid wealth invested in the j-th liquid risky asset.

$$L(t)+\sum_{i=m_1}^{m} A_i Q_i(t)+\sum_{i=m+1}^{M}(1-\lambda_i)Q_i(t)\geq 0 \text{ almost surely for all } t\geq 0, \tag{8}$$

where $I_{m_1}, \cdot\cdot\cdot, I_m$ are the illiquid assets that can be liquidated. Constraint (8) says that when all the assets are converted to liquid assets, the total value cannot become negative.

I define the value function V_t of the agent as the maximum possible utility obtainable with feasible consumption and portfolio strategies:

$$V_t=\max U_t \tag{9}$$

Under the above fomulation of the problem the value function can be expressed as a function of liquid wealth L_t and $Q_1, Q_2, \cdot\cdot\cdot, Q_M$, i.e.,

$$V_t=V(L(t), Q_1(t), \cdot\cdot\cdot, Q_M(t)). \tag{10}$$

Now I define the marginal price $pi(t)$[1] is a function of the i-th illiquid asset by

$$pi(t)\equiv\frac{V_{Q_i}(L(t), Q_1(t), \cdot\cdot\cdot, Q_M(t))}{V_L(L(t), Q_1(t), \cdot\cdot\cdot, Q_M(t)}, \tag{11}$$

1) The marginal price is a function of $L(t), Q_1(t), \cdot\cdot\cdot, Q_M(t)$, but in order to simplyfy the notation I will denote it by $pi(t)$.

where subscripts L and Q_i denote partial derivatives with respect to these variables. The implicit total wealth $W(t)$ of the agent is defined to be

$$W(t) \equiv L(t) + p_1(t)Q_1(t) + \cdot\cdot\cdot + p_M(t)Q_M(t). \tag{12}$$

The unit-wealth value function $q(t)/(1-\gamma)$ is defined to be

$$\frac{q(t)}{1-\gamma} \equiv \frac{V(L(t), Q_1(t), \cdot\cdot\cdot, Q_M(t))}{W(t)^{1-\gamma}}, \tag{13}$$

i.e., it gives the value of the value function when the implicit total wealth is equal to one.

The value function can now be represented as a function of implicit total wealth in the following form:

$$V(L(t), Q_1(t), \cdot\cdot\cdot, Q_M(t)) = \frac{q(t)}{1-\gamma} W(t)^{1-\gamma}. \tag{14}$$

The usefulness of the above representation is illustrated by the following envelop theoremlike result:

Proposition 1: The first-order partial derivatives of V take the following form:

$$V_L(L(t), Q_1(t), \cdot\cdot\cdot, Q_M(t)) = q(t)W(t)^{-\gamma}$$

$$V_{Q_i}(L(t), Q_1(t), \cdot\cdot\cdot, Q_M(t)) = p_i(t)q(t)W(t)^{-\gamma} \quad \text{for} \quad i = 1, 2, \cdot\cdot\cdot, M.$$

Proof: See the appendix.

Namely, when we take the first-order derivatives the functions $p(t), \cdots, p_M(t)$ and $q(t)$ can be regarded as if they were constants.

The agent does not trade illiquid financial assets continuously, otherwise the trading costs incurred would be infinite. When the investor does not trade illiquid assets, the Hamilton-Jacobi-Bellman equation which describes optimal balancing of current and futures consumption takes the following form:

$$\delta V_t = \max\left[v(C_t) + \frac{E_t[dV]}{dt} \right] \tag{15}$$

where the maximum is taken over all consumption and portfolio strategies which satisfy constraint (7). When the investor does not trade illiquid assets, the above HJB equation implies the following optimal policies:

Proposition 2:

(i) Optimal consumption takes the from

$$C(t) = \begin{cases} q(t)^{\frac{-1}{\gamma}} W(t) & \text{if } \gamma \neq 1 \\ \delta W(t) & \text{if } \gamma = 1 \end{cases}$$

(ii) Optimal investment in liquid assets takes the form

$$\omega_t = \frac{1}{\gamma_L(t)} (\Sigma \Sigma^*)^{-1} (\mu - \gamma 1) - \sum_{i=1}^{M} \left(\frac{pi(t) Q_i(t)}{L(t)} - \frac{\partial pi(t)}{\partial L} \frac{Q_i(t)}{\gamma L(t)} \right) \tau (\Sigma^*)^{-1} \Xi_i$$

where

$$\gamma_L(t)\equiv\left[\ \gamma+\frac{\partial pi(t)}{\partial L}Q_i(t)\right]\frac{L(t)}{W(t)}$$

and

$$\omega_t\equiv(\omega_1(t),\cdot\cdot\cdot,\omega_N(t))^*\ \ \Sigma\equiv(\sigma_{jk})_{j,k=1}^{N}\ \ \mu\equiv(\mu_1,\cdot\cdot\cdot,\mu N)^*$$

$$1\equiv(1,\cdot\cdot\cdot,1)^*\ \ \Xi_i\equiv(\rho_{ki})_{k=1,}\cdot\cdot\cdot,N,\quad \text{for } i=1,\cdot\cdot\cdot,M$$

where * denotes the transpose of a matrix, $\omega_j(t)$ is the proportion of liquid wealth invested in the j-th liquid risky asset, and ρ_{ki} is the correlation between $dW_k(t)$ and $dB_i(t)$.

Proof: See the appendix.

Proposition 2 (i) says that consumption is a proportion of the implicit total wealth, where the proportion is constant if $\gamma=1$ and time varying if $\gamma\neq1$. This proposition justifies the use of the marginal price $dpi(t)$ to evaluate the value of the illiquid asset I_i. I will call the value $pi(t)Q_i(t)$ the marginal value of the illiquid asset.[2)]

The vector $(\Sigma\Sigma^*)^{-1}(\mu-\gamma 1)$ descrbes the mean-variance effcient (i.e., the tangency portfolio in the space of all liquid risky assets) and the vector $-\tau(\Sigma^*)^{-1}\Xi_i$ is the portfolio which has the highest correlation with $Q_i(t)$ (i.e., the hedging portfolio that hedges risk in the illiquid asset L_i). Proposition 2 (ii) says that the optimal investment in the meanvariance effcient portfolio is

2) I will also discuss the average values insection 3.

determined by two considerations: first, the agent calculates the implicit total value of wealth and wants to invest an optimal proportion of the implicit total value in each liquid risky asset, and second, the agent appropriately adjusts risk tolerance taking into account that he or she owns illiquid assets. The term $L(t)/W(t)$ in the definition of $\gamma_L(t)$ captures the first consideration. The term $\partial pi(t)/\partial L)Q_i(t)$ captures adjustment to the risk aversion coefficient. The following proposition shows that this term is nonnegative.

Proposition 3: For $i=1, \cdot\cdot\cdot, M$, $pi(t)$ is non-decreasing in L, i.e.,

$$\frac{\partial pi(t)}{\partial L} \geq 0$$

Proof: See the appendix.

Proposition 3 implies that the effective relative risk aversion coefficient used in the investment in the mean-variance efficient portfolio is in general greater than the coefficient of relative risk aversion γ implied by the felicity function. Namely, the agent's risk tolerance declines in the face of the illiquid assets.

Proposition 3 also says that the marginal value of an illiquid asset increases as liquid assets increase. As liquid assets increase, the negative effects idiosyncratic risks in the illiquid assets on the agent's welfare, therefore, the marginal value increases.

3. Examples

In this section Iconsider two particular examples to which the general results in the previous section applies. In Example1, I consider a consumption and investment problem in which there is one illiquid financial asset with large trading costs. In Example 2, I consider a consumption and investment problem of an agent who owns an illiquid asset which is either non-traded or can be liquidated only in its entirety. In the examples there is only illiquid asset and I will denote Q_1, A_1, and λ_1 by Q, A, and λ dropping the subscript.

Since there is only one illiquid asset in the examples, the results in the previous section take simpler forms. Namely, I define the marginal price p of the non-traded asset by

$$p \equiv \frac{V_Q(L, Q)}{V_L(L, Q)}. \tag{16}$$

Since the value function is homogeneous of degree $1-\gamma$, p is homogeneuous of degree zero, and therefore, it can be written as a function of the ratio

$$z \equiv \frac{L}{Q}. \tag{17}$$

Since this ratio z will occur very frequently in the paper, I will give a name to z and call it the liquidity ratio. Then, Proposition 3 in the previous section can be stated as

$$p'(z) \geq 0, \tag{18}$$

Proposition 2 now takes the following form in this case:

Proposition 4: (i) Optmal consumption takes the form

$$C(t)=\begin{cases} q(t)^{\frac{-1}{\gamma}}W(t) & \text{if } \gamma\neq 1 \\ \delta W(t) & \text{if } \gamma=1 \end{cases}$$

(ii) When short-selling constraints are slack, optimal investment in liquid assets takes the form

$$\omega_t=\frac{1+p(z)/z}{\gamma+p'(z)}(\Sigma\Sigma^*)^{-1}(\mu-\gamma 1)-\frac{\gamma p(z)-p'(z)}{\gamma+p'(z)}\tau(\Sigma)^{-1}\Xi$$

In these examples I will consider also the average value of the illiquid asset, which is defined as follows: the average price P of the illiquid asset is defined to be such that the agent is indifferent between owning the illiquid asset and replacing it by a liquid asset with value PQ (if he or she is given such an opportunity), and the average value of the illiquid asset is defined to be PQ. The agent's value function in the absence of the illquid asset is given by

$$V(L,0)=\frac{K^{-\gamma}}{1-\gamma}L^{1-\gamma}. \tag{19}$$

Therefore, PQ satisfies

$$\frac{K^{-\gamma}}{1-\gamma}(L+PQ)^{1-\gamma}=V(L,Q) \tag{20}$$

By homogeneity of the value function P is a function of the liquidity ratio z.

Example 1: In this example I will consider the case where there is only one illiquid asset which can be sold or bought both in small and large units. The face value and number of shares of the asset which is owned by the agent are denoted by $Q(t)$ and $J(t)$. When the agent wants to buy x additional shares of the illiquid asset, he or she needs to pay $xQ(t)/J(t)$ in liquid assets. When the agent sells x shares of the illiquid asset, however, he or she will get only $(1-\lambda)Q(t)/J(t)$ in liquid assets. A similar problem without portfolio selection of liquid assets has been considered by Constantinides (1986) and Davis and Norman (1990).

Trading costs for stocks of large well-recognized companies are relatively small nowadays. The average bid-ask spread for large company stocks is about 0.69% of the prices of the stocks (Stoll and Whaley (1983)). A turnaround commission for these stocks can be smaller than 0.2% for large investors and this commission can be lower than 0.05% if investors use futures contracts to rebalance their portfolios. Trading stocks of small unrecognized companies, however, is very costly. The bid and ask spreads of small NASDAQ stocks are well above 10% of the averages of their bid and ask prices and can reach as high as 35% (Lamoureux and Sanger 1989). This implies that regarding stocks of large companies as liquid and stocks of small companies as illiquid can possibly by a reasonable approximation to the modern financial markets.

The marginal value of the illiquid asset cannot be lower than the value for which it can be sold in small units or cannot be greater than the value for which it can be bought in small units. Therefore,

$$1-\lambda \leq p(z) \leq 1. \tag{21}$$

Trading occurs only when one of the inequalities holds as equality. Since $p(z)$ is a monotone increasing function, inequality (18) implies that there is an interval [a, b] such that

$$p(a)=1-\lambda, \quad p(b)=1, \quad \text{and} \quad 1-\lambda < p(z) < 1 \quad \text{for} \quad a < z < b \tag{22}$$

Trading of the illiquid asset does not occur when the liquidity ratio $z=Q/L$ lies inside the interval, because the investor's marginal valuation of the illiquid asset does not justify the trade. The investor buys the illiquid asset whenever the liquidity ratio z becomes greater than b and sells it whenever the liquidity ratio becomes smaller than a. The amount which is bought or sold is such that the liquidity ratio z is pulled back to the nearest boundary of the interval, since trading costs are proportional to the amount traded (there is no fixed cost) and therefore it is not optimal to adjust the liquidity ratio more than justified by the subjective marginal value of the illiquid asset (if the ratio were adjusted to a value inside the interval then the marginal value of the illiquid asset would be greater than $1-\lambda$ or lower than 1 and the adjustment would violate the necessary condition for utility maximization).

When buying (or selling) the illiquid asset, the marginal rate of substitution $p(z)$ should be equal to $1+\lambda$ (or $1-k$) in order to justify the transaction, so the function $p(z)$ satisfies

$$p(z)=\begin{cases} 1-\lambda & \text{if } z\leq a \\ 1 & \text{if } z\geq b \end{cases} \tag{23}$$

By the smooth pasting condition (see Dixit (1991)), $p(z)$ and $q(z)$ are continuously differentiable also at the boundary points a and b of the interval for non-trading and the Bellman equation is satisfied at these boundaries. In paricular, $p(z)$ is constant over $(0, a)$ and (b, ∞) by equation (23), therefore,

$$p'(a) = 0, \qquad p'(b) = 0 \tag{24}$$

Example 2: In this example I consider a consumption and investment problem of an agent who owns an asset which is either non-traded or can be liquidated only in its entirety. If the asset can be liquidated, I will call it a proprietorship.

The illiquid asset generates cash flows at the rate $Q(t)$ during an infinitesimal time period $(t, t+dt)$. Confirming to the assumption in the previous section, $xQ(t)$ evolves according to

$$\frac{dQ(t)}{Q(t)} = \nu\, dt + \tau\, dB(t) \tag{25}$$

Koo (1994b) has shown that the following properties of $p(z)$:

Proposition 5: (i) If $\tau > 0$ and $B(t)$ is not spanned by $W_1(t), \cdot\cdot\cdot, W_N(t)$, then $p(z)$ is strictly increasing in z. (ii) If $\mu_Q \equiv \tau\, \theta^* \Xi + \gamma - \nu > 0$, where $\theta \equiv \Sigma(\mu - \gamma 1)$, then

$$\lim_{z\to\infty} p(z) = \frac{1}{\mu_Q} \qquad \lim_{z\to\infty} q(z) = K^{-\gamma}$$

where

$$K \equiv \gamma + \frac{\delta - \gamma}{\gamma} - \frac{1}{2}\frac{1-\gamma}{\gamma^2}\theta^*\theta.$$

Proposition 5 says that generally the marginal value $p(z)Q$ of the non-traded asset is smaller than Q/μ_Q. In fact, the latter value satisfies

$$\frac{Q(t)}{\mu_Q} \equiv E_t \int_t^{\infty} \frac{\zeta(s)}{\zeta(t)} Q(s)ds \tag{26}$$

where

$$\zeta(t) \equiv \exp\left\{-rt - \theta^* W(t) - \frac{1}{2}\theta^*\theta t\right\}. \tag{27}$$

Namely, Q/μ_Q is the present value of future cash flows in which the risk premium for risk orthogonal to risks in liquid financial assets is not priced. Proposition 5 says that the marginal value of the non-traded asset approaches this present value.

The following proposition describes properties of the average price P.

Proposition 6: (i) The average price is a non-decreasing function of z. Further, if $\tau \neq 0$ and $B(t)$ is not spanned by $W_1(t), \cdots, W_N(t)$, then $P(z)$ is a strictly increasing function. (ii) The average value of the non-traded asset is not smaller than the marginal value. Further, if $\tau \neq 0$ and $B(t)$ is not spanned by

$W_1(t), \cdots, W_N(t)$, then the average value is larger than the marginal value.

Proof: See the appendix.

If the asset is a proprietorship, then the agent will liquidate it when the average value of the non-traded asset is equal to the liquidation value. Consumption is in general discontinuous at this moment:

Proposition 7: Suppose that $\tau \neq 0$ and $B(t)$ is not spanned by $W_1(t), \cdots, W_N(t)$. When the agent liquidates the proprietorship, optimal consumption jumps upward.

Proof: See the appendix.

While the agent owns the proprietorship, the agent saves more than he would in the absence of proprietorship in order to prepare for unexpected decline in the cash flows from the proprietorship (see Kimball (1990) for discussion of this precautionary motive for saving). Therefore, when the agent liquidates the proprietorship, such precautionary motive disappears and consumption is revised upward.

4. Conclusion

In this paper I have defined the marginal values of illiquid asset in the framework of a consumption and investment model, and studied how optimal consumption and investment policies are related to the marginal values. I have applied the results of the general study to concrete examples and shown how the existence of illiquid assets influences investors' risk taking in financial markets as well as their hedging activity.

The study shows that investors' activity in financial markets can be significantly affected by their ownership of illiquid assets such as human capital, private businesses, and small company stocks. In particular, this implies that it would be impossible to understand investors' activity in the financial markets if the existence of highly illiquid assets in investors' portfolios are ignored.

The influence of habit formation and durable consumption on investors' risk taking studied by Constantinides (1989), Grossman and Laroque (1990) and Hindy and Huang (1993) can also be understood within the framework of this paper; habit formation and durable consumption both mean investors' satisfaction depend on past consumption as well as current consumption, and if past consumption is regarded as an illiquid asset (either good or bad) then the result that investors' risk taking is affected by habit formation or durable consumption can be interpreted as a particular case of the result in this paper, namely, investors' risk aversion is influenced by the existence of an illiquid asset.

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