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WORMHOLES IN SCALAR-TENSOR THEORIES OF GRAVITY

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ABSTRACT

Wormhole solutions of general theory of relativity are known to violate energy conditions. We have considered the possibility of having wormhole solutions in Brans-Dicke theory which is the prototype of scalar-tensor theories of gravity.

The metric

$$d\tau^2 = dt^2 - dr^2 - (r^2 + r_o^2)(d\theta^2 + sin^2\theta d\phi^2) \qquad (1)$$

is probably the simplest example for a 3+1 dimensional wormhole geometry. In such a spacetime, non-trivial 2-spheres have a minimum area $4\pi r_o^2$ and an equatorial circle has a minimum circumference $2\pi r_o$, while the radial coordinate can contineously vary from $-\infty$ to 0 to $+\infty$.
The important question is whether a plausible energy-momentum tensor can support a wormhole geometry. For the metric (1), Einstein's equations with a T^μ_ν corresponding to an anisotropic fluid

$$T^\mu_\nu = diag(-\rho, p_r, p_t, p_t), \qquad (2)$$

yield

$$\rho(r) = -T^t_t = -\frac{r_o^2}{8\pi G(r_o^2 + r^2)^2}, \qquad (3)$$

$$p_r(r) = T^r_r = \rho(r), \qquad (4)$$

$$p_t(r) = T^\theta_\theta = T^\phi_\phi = -\rho(r). \qquad (5)$$

This, however, is an unusual fluid with negative energy density and radial pressure and thus violating energy conditions. Although the violation of classical energy conditions is generic in more general wormhole metrics, these topogical objects can not be completely ruled out, if we consider quantum field theoretical arguments which allow exotic matter (see e.g. Epstein et al. 1965).
The Brans-Dicke field equations which result from the action (Weinberg, 1972)

$$\mathcal{A} = \frac{1}{16\pi}\int(-\varphi R - \frac{\omega}{\varphi}\varphi_{;\mu}\varphi^{;\mu} + 16\pi\mathcal{L}_m)\sqrt{g}d^4x \qquad (6)$$

read

$$\varphi G_{\mu\nu} = -8\pi T_{m\mu\nu} - \frac{\omega}{\varphi}(\varphi_{;\mu}\varphi_{;\nu} - \frac{1}{2}g_{\mu\nu}\varphi_{;\rho}\varphi^{\rho}_{;}) - \varphi_{;\mu;\nu} + g_{\mu\nu}\Box^2\varphi, \qquad (7)$$

and

$$\Box^2\varphi = \frac{8\pi}{2\omega + 3}T^\lambda_{m\lambda}. \qquad (8)$$

In these equations, φ is a scalar field, and ω is the dimensionless B-D coupling constant. Roughly speaking, GR is the $\omega \to \infty$ limit of B-D theory.
It is interesting to note that a 4+n dimensional Kaluza-Klein spacetime with the metric γ_{AB} ($A, B = 1, 2, ..., 4+n$) can be decomposed as $\gamma_{AB} = g_{\mu\nu} + \varphi_{ab}$ ($\mu, \nu = 1, 2, 3, 4,$ *and* $a, b = 5, 6, ..., 4+n$), and the Lagrangian is expressible (after integrating over the extra dimensions) as (Cho 1992)

$$\mathcal{L} = -\sqrt{g}\left[\varphi R + \frac{\omega}{\varphi}\partial^\mu\varphi\partial_\mu\varphi + other \;\; terms\right]. \qquad (9)$$

In this Lagrangian,

$$\omega = -\frac{n-1}{n}, \qquad and \qquad \varphi = \frac{1}{16\pi G}|det\varphi_{ab}|^{1/2}. \qquad (10)$$

Note that $\omega < 0$ for $n > 1$.
Under the conformal transformations

$$g_{\mu\nu} \to \bar{g}_{\mu\nu} = (\frac{\varphi}{\varphi_o})^{1-\alpha}g_{\mu\nu} \qquad (11)$$

in which φ_o and α are constants, equations (7) and (8) transform into

$$\bar{G}_{\mu\nu} = -\frac{8\pi(\frac{\varphi}{\varphi_o})^{-\alpha}}{\varphi_o}\bar{T}_{m\mu\nu} - \frac{1}{2}(2\omega + 3 - \alpha^2)\Lambda_{;\mu}\Lambda_{;\nu} + \frac{1}{4}(2\omega + 3 + \alpha^2)\bar{g}_{\mu\nu}\Lambda_{;\lambda}\Lambda^\lambda_; - \alpha(\Lambda_{;\mu;\nu} - \bar{g}_{\mu\nu}\Box^2\Lambda) \qquad (12)$$

and

$$\Box^2\Lambda + \alpha\Lambda_{;\lambda}\Lambda^\lambda_; = \frac{8\pi(\frac{\varphi}{\varphi_o})^{-\alpha}}{\varphi_o(2\omega + 3)}\bar{T}^\mu_{m\mu}, \qquad (13)$$

in which $\Lambda = ln(\varphi/\varphi_o)$.
In these equations the same coordinates x^μ are used, while all metric coefficients used to compute geometrical objects like $\bar{G}_{\mu\nu}$ and to raise and lower indices are $\bar{g}_{\mu\nu}$'s.
We obviously have

$$\bar{g}^{\mu\nu} = (\frac{\varphi}{\varphi_o})^{\alpha-1}g^{\mu\nu}, \qquad (14)$$

and

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = (\frac{\varphi}{\varphi_o})^{\alpha-1}\bar{g}_{\mu\nu}dx^\mu dx^\nu. \qquad (15)$$

These transformations can be used to obtain exact solutions B-D equations (Askari 1994).
Here, we limit ourselves to an exact spherically symmetric wormhole solution. Let us choose $\alpha = 0$ in (14) and (15). This is Dicke's choice in which mass varies while the gravitational coupling $\bar{G}$ remains constant. Equations (14) and (15) with $T_M^{\mu\nu} = 0$ then read

$$\bar{G}_{\mu\nu} = -\frac{2\omega+3}{2}\Lambda_{;\mu}\Lambda_{;\nu} + \frac{2\omega+3}{4}\bar{g}_{\mu\nu}\Lambda_{;\alpha};^{\alpha}, \tag{16}$$

and

$$\Box^2\Lambda = 0. \tag{17}$$

For a spacetime with the metric

$$d\bar{s}^2 = \bar{g}_{\mu\nu}dx^\mu dx^\nu \tag{18}$$

$$\bar{g}_{\mu\nu} = B(r)dt^2 - A(r)dr^2 - r^2 d\Omega^2 \tag{19}$$

Equations (16) and (17) are expressible in the form

$$\bar{R}_{\mu\nu} = -8\pi G\psi_{;\mu}\psi_{;\nu}; \tag{20}$$

and

$$\Box^2\psi = 0, \tag{21}$$

in which

$$\psi(r) = \sqrt{\frac{2\omega+3}{16\pi G}}\Lambda(r). \tag{22}$$

These equations yield

$$\frac{d}{dr}ln(AB) = 8\pi G r\psi'^2(r), \tag{23}$$

$$\frac{d}{dr}ln\sqrt{\frac{B}{A}} = \frac{1}{r}(A-1). \tag{24}$$

and

$$\frac{d}{dr}\left[r^2\sqrt{\frac{B}{A}}\psi'(r)\right] = 0. \tag{25}$$

It can be shown that an exact solution to these equations is given as

$$B(r) = 1, \tag{26}$$

$$A(r) = \frac{1}{1+\frac{4\pi GQ^2}{r^2}}. \tag{27}$$

and

$$\psi(r) = \frac{1}{\sqrt{4\pi G}}sinh^{-1}\frac{\sqrt{4\pi G}Q}{r}. \tag{28}$$

In these equations, Q is a constant of integration (let us call it *scalar charge*). In the case $2\omega + 3 < 0$, we obtain

$$B(r) = 1, \tag{29}$$

$$A(r) = \frac{1}{1-\frac{4\pi GQ^2}{r^2}}, \tag{30}$$

$$\psi(r) = \frac{1}{\sqrt{4\pi G}}tan^{-1}\frac{\sqrt{4\pi G}|Q|}{\sqrt{r^2 - 4\pi GQ^2}}. \tag{31}$$

The metric -in this case- is that of a wormhole. To see this, we conformally invert this solution of GR into the corresponding solution of B-D equations

$$ds^2 = (\frac{\varphi_o}{\varphi})\left[dt^2 - \frac{dr^2}{1-\frac{4\pi GQ^2}{r^2}} - r^2 d\omega^2\right], \tag{32}$$

in which

$$\varphi(r) = \varphi_o exp\left[\frac{2}{\sqrt{-(2\omega+3)}}tan^{-1}\frac{\sqrt{4\pi G}|Q|}{\sqrt{r^2-4\pi GQ^2}}\right] \tag{33}$$

It is evident that $\varphi(r) \rightarrow \varphi_o$ as $r \rightarrow \infty$, and the radial coordinate r has a minimum value of $r_o = \sqrt{4\pi G}|Q|$. The coordinate transformation $r^2 = \tilde{r}^2 + r_o^2$, turns the metric (32) into

$$ds^2 = \frac{\varphi_o}{\varphi(\tilde{r})}\left[dt^2 - d\tilde{r}^2 - (\tilde{r}^2 + r_o^2)d\Omega^2\right]. \tag{34}$$

with

$$\varphi(\tilde{r}) = \varphi_o exp\left[\frac{2}{\sqrt{-(2\omega+3)}}tan^{-1}\frac{r_o}{\tilde{r}}\right] \tag{35}$$

In order to relax the ω constraint, we can include a matter energy-momentum tensor with suitable properties, or extend the theory by adding a self-interacting potential for the scalar field, as in the induced gravity theory of Zee (1981). This route is explored by Accetta et. al. (1990).

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