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GEOMETRIC CONSIDERATION OF DUALITY IN MULTIOBJECTIVE OPTIMIZATION WITH SET FUNCTIONS

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1. Introduction.

In ordinary scalar convex optimization, the Lagrange multiplier theorem asserts the existence of a supporting hyperplane for the epigraph of the primal map[5]. On the other hand, in multiobjective optimization, the corresponding theorem [7, Theorem 2.3] implies the existence of a conical variety (i.e., a translation of a cone) which supports the epigraph of primal map. In this paper, we show that a similar assertion is true for the multiobjective programming problem with set functions.

2. Multiobjective Programming Problem with Set Functions.

Let $(X, \mathcal{A}, \mu)$ be a finite, atomless measure space and $L^1(X, \mathcal{A}, \mu)$ be separable. Then,by considering characteristic function χ_Ω of Ω in $\mathcal{A}$, we can embed $\mathcal{A}$ into $L^\infty(X, \mathcal{A}, \mu)$. In this setting for $\Omega, \Lambda \in \mathcal{A}$, and $\alpha \in I = [0, 1]$, there exists a sequence, called *a Morris sequence*, $\{\Gamma_n\} \subset \mathcal{A}$ such that

$$\chi_{\Gamma_n} \xrightarrow{w^*} \alpha\chi_\Omega + (1-\alpha)\chi_\Lambda,$$

where $\xrightarrow{w^*}$ denotes the *weak**– convergence of elements in $L^\infty(X, \mathcal{A}, \mu)$ [6].

A subfamily $\mathcal{S}$ is said to be *convex* if for every $(\alpha, \Omega, \Lambda) \in I \times \mathcal{S} \times \mathcal{S}$ and every Morris sequence $\{\Gamma_n\}$ associated with $(\alpha, \Omega, \Lambda)$ in $\mathcal{A}$, there exists a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$ in $\mathcal{S}$. In [1], if $\mathcal{S} \subseteq \mathcal{A}$ is convex, then the *weak**–closure $cl(\mathcal{S})$ of $\chi_\mathcal{S}$ in $L^\infty(X, \mathcal{A}, \mu)$ is the *weak**–closed convex hull of $\chi_\mathcal{S}$, and $\overline{\mathcal{A}} = \{f \in L^\infty : 0 \le f \le 1\}$.

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DEFINITION 2.1. *Let $\mathcal{S}$ be a convex subfamily of $\mathcal{A}$. Let K be a convex cone of R^n. A set function $H : \mathcal{S} \to R^n$ is called K-convex, if given $(\alpha, \Omega_1, \Omega_2) \in I \times \mathcal{S} \times \mathcal{S}$ and Morris-sequence $\{\Gamma_n\}$ in $\mathcal{A}$ associated with $(\alpha, \Omega_1, \Omega_2)$, there exists a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n)$ in $\mathcal{S}$ such that*

$$\limsup_{k\to\infty} H(\Gamma_{n_k}) \leqq_K \alpha H(\Omega_1) + (1-\alpha)H(\Omega_2),$$

where lim sup is taken over each component.And $x <_K y$ denotes $y - x \in int(K)$,$x \leq_K y$ denotes $y - x \in K \setminus \{0\}$, and $x \leqq_K y$ denotes $y - x \in K$.

DEFINITION 2.2. *A set function $H = (H_1, H_2, \ldots, H_n) : \mathcal{S} \to R^n$ is called weak*-continuous on $\mathcal{S}$ if for each $f \in cl(\mathcal{S})$ and for each $j = 1, 2, \ldots, n$, the sequence $\{H_j(\Omega_k)\}$ converges to the same limit for all $\{\Omega_k\}$ with $\chi_{\Omega_k} \xrightarrow{w^*} f$.*

Now multiobjective programming problem with set functions can be described as follows:

$$\text{(P)} \qquad \begin{aligned} &Min_D \quad F(\Omega) \\ &\text{subject to } \Omega \in \mathcal{S} \\ &\text{and } G(\Omega) \leqq_Q \mathbf{0}, \end{aligned}$$

which has been defined as the problem finding all feasible efficient $D-$ or properly efficient $D-$solution with respect to the pointed closed convex cones D and Q of Euclidean spaces R^p and R^m with nonempty interiors, D^o and Q^o, respectively. That is, letting $\mathcal{S}' = \{\Omega \in \mathcal{S} : G(\Omega) \leqq_Q 0\}$,we want to find $\Omega^* \in \mathcal{S}'$ such that

$$(F(\mathcal{S}') - F(\Omega^*)) \cap (-D) = \{\mathbf{0}\}, \quad \emptyset \text{ if } \mathbf{0} \notin D$$

or

$$cl(p(F(\mathcal{S}') + D - F(\Omega^*))) \cap (-D) = \{\mathbf{0}\}, \emptyset \text{ if } \mathbf{0} \notin D,$$

where the set $p(S) = \{\alpha y \colon \alpha > 0, y \in S\}$ is the projecting cone for a set $S \subset R^p$. We denote the set of efficient D-solutions by $\mathcal{E}(F(\mathcal{S}'), D)$ and the set of properly efficient D-solutions by $\mathcal{PE}(F(\mathcal{S}'), D)$.

For the primal problem (P), we assume that $F : \mathcal{S} \to R^p$, $G : \mathcal{S} \to R^m$ are $D-$convex, $Q-$convex, respectively and *weak**-continuous. Under this assumptions we have the Lagrange multiplier theorem as in usual multiobjective optimization problems. The set of $p \times m$ matrices $\{M \in R^{p\times m} \colon MQ \subset D\}$ is denoted by $\mathcal{M}$.

THEOREM 2.3. [3] *Let Ω^* be a properly efficient $D-$solution to the problem (P). If there is $\Omega_o \in \mathcal{S}$ such that $G(\Omega_o) <_Q \mathbf{0}$, then there exists $M^* \in \mathcal{M}$ such that*

(1) $F(\Omega^*) \in Min_D\{F(\Omega) + M^*G(\Omega) : \Omega \in \mathcal{S}\}$

(2) $M^*F(\Omega^*) = \mathbf{0}$.

In fact, $F(\Omega^*) \in Min_D cl(F(\Omega) + M^*G(\Omega) \colon \Omega \in \mathcal{S})$.

The generalized Slater's constraint qualification in Theorem 2.3 that there exists $\Omega_o \in \mathcal{S}$ such that $G(\Omega_o) <_Q \mathbf{0}$ is assumed in the sequel.

The primal problem (P) is embedded into a family of perturbed problems:

$$(\mathrm{P}_u) \qquad \begin{aligned} &Min_D \quad F(\Omega) \\ &\text{subject to} \quad \Omega \in \mathcal{S} \\ &\text{and} \quad G(\Omega) \leqq_Q u. \end{aligned}$$

We denote by $\mathcal{S}(u)$ the set $\{\Omega \in \mathcal{S} : G(\Omega) \leqq_Q u\}$, and by $Y(u)$ the set $F(\mathcal{S}(u))$.

DEFINITION 2.4. *Perturbed (or primal) maps are defined on R^m by*

$$W(u) = Min_D F(\mathcal{S}(u))$$

$$\text{and} \qquad \overline{W}(u) = Min_D cl(F(\mathcal{S}(u)))$$

The original problem (P) can be therefore regarded as determining $F^{-1}(W(\mathbf{0})) \cap \mathcal{S}$. However, more satisfactory results are obtained if $\overline{W}$ is used instead.

For each $M \in \mathcal{M}$, we define certain maps for (P) on $\mathcal{M}$ by

$$\Phi(M) = Min_D\{F(\Omega) + MG(\Omega) : \Omega \in \mathcal{S}\}$$

$$\overline{\Phi}(M) = Min_D \ cl(\{F(\Omega) + MG(\Omega) : \Omega \in \mathcal{S}\})$$

The map Φ and $\overline{\Phi}$ are called dual maps for (P).

REMARK 2.5.

(1) $MG(\cdot) : \mathcal{S} \to R^p$ *is* $D-$*convex on* $\mathcal{S}$.

(2) $L(\cdot, M) = F(\cdot) + MG(\cdot)$ *is* $D-$*convex and* w^*-*continuous.*

(3) $cl(\{F(\mathcal{S}) + MG(\Omega)\})$ *is D-convex subset of* R^p

(4) *For each* $M \in \mathcal{M}$, *we have*

$$cl(\{L(\Omega, M) : \Omega \in \mathcal{S}\}) + D = \overline{\Phi}(M) + D,$$

since $cl(\{L(\Omega, M) : \Omega \in \mathcal{S}\})$ *is compact and* $D-$*convex.*

(5) *For any* u *with* $\mathcal{S}(u) \neq \emptyset$, $[clY(u)] + D = \overline{W}(u) + D$.

The relationship between the primal map $\overline{W}$ and the dual map $\overline{\Phi}$ now can be established.

THEOREM 2.6. [4] *For any* $M \in \mathcal{M}$, *the following equalities hold.*

$$\overline{\Phi}(M) = Min_D \bigcup_{u\in\zeta}(\overline{W}(u) + Mu) = Min_D \bigcup_{u\in\zeta^o}(\overline{W}(u) + Mu)$$

where $\zeta = \{u \in R^m \colon \mathcal{S}(u) \neq \emptyset\}$ *and* $\zeta^o = \{u \in R^m \colon \{\Omega \in \mathcal{S} \colon G(\Omega) <_Q u\} \neq \emptyset\}\}$.

COROLLARY 2.7. *If* Ω^* *is a properly efficient D-solution to the problem (P) with generalized Slater's constraint qualification, then there exists an* $M^* \in \mathcal{M}$ *such that*

$$F(\Omega^*) \in \overline{\Phi}(M^*) \cap \Phi(M^*) \subset Min_D[\textstyle\bigcup_{u\in\zeta}(\overline{W}(u) + M^*u)].$$

proof. The proof is an immediate consequence of Theorems 2.3 and 2.6.

3. A Supporting Conical Variety.

All assumptions on F, G, D and Q of section 1 and 2 are inherited. Furthermore, D is assumed to be a polyhedral convex cone. Then D is the set of all solutions to some finite system of homegenous weak linear inequalities. Hence there exists an $s \times p$ matrix U_1 such that

$$D = \{y \in R^p : U_1 \cdot y \geqq 0\}, \tag{1}$$

where the row vectors of U_1 are generators of D^o. Since D is pointed, the $s \times p$ matrix U_1 has the full rank p. Here $\mathcal{M} = \{M \in R^{m\times p} : MQ \subset D\}$.

DEFINITION 3.1. *The D-epigraph of W is defined by*

$$D\text{-epi } W = \{(u, y) \in R^m \times R^p : u \in \zeta, y \in W(u) + D\}.$$

Here $W(u) = Min_D Y(u)$ and $\zeta = \{u : \mathcal{S}(u) \neq \emptyset\}$ were introduced in section 2. The $D-$epigraph of $\overline{W}$ is defined similarly. Unlike the case of ordinary convex multiobjective optimization problem, D-epi W may not be a closed convex set in $R^m \times R^p$. But we can see that $cl(D\text{-epi } W)$ is convex. The following lemma is similar to Lemma 2.4 [7].

LEMMA 3.2. *For a given $M \in \mathcal{M}$, the following conditions are equivalent :*
(i) $F(\Omega) + MG(\Omega) \not\leq_D F(\Omega^*) + MG(\Omega^*)$ *for any* $\Omega \in \mathcal{S}$ *;*
(ii) $U_1F(\Omega) + U_1MG(\Omega) \not\leq_{R^s_+} U_1F(\Omega^*) + U_1MG(\Omega^*)$ *for any* $\Omega \in \mathcal{S}$.

Proof. Let $Y = \{F(\Omega) + MG(\Omega) : \Omega \in \mathcal{S}\}$. Let $y = F(\Omega) + MG(\Omega)$ and $y^* = F(\Omega^*) + MG(\Omega^*)$. Then $y^* \notin \mathcal{E}(Y, D)$ if and only if there is a $y \in Y$ such that $y^*\text{-}y \in D \setminus \{0\}$. This is so since $Ker\ U_1 = \{0\}, y^* - y \in D \setminus \{0\}$ if and only if $U_1(y^* - y) \geq 0$. That is, $U_1y^* \geq U_1y$ if and only if $U_1y^* \notin \mathcal{E}(U_1y, R^s_+)$.

Now we consider the above lemma for D-epi W.

LEMMA 3.3. *For $M \in \mathcal{M}$, the following are equivalent:*
(i) For some $(u^*, y^*) \in D$-epi W and all $(u, y) \in D$-epi W,

$$y + Mu \not\leq y^* + Mu^*;$$

(ii) For some $\Omega^* \in \mathcal{S}$ and all $\Omega \in \mathcal{S}$,

$$F(\Omega) + MG(\Omega) \not\leq F(\Omega^*) + MG(\Omega^*).$$

Proof. It is similar to that of [7, Lemma 2.5].

Following Nakayama [7], we define a supporting conical variety as follows: For $M \in \mathcal{M}$, let $U_2 = MU_1$ and define a cone K in $R^m \times R^p$ by

$$K = \{(u, y) : U_1y + U_2u \leqq 0, u \in R^m, y \in R^p\}. \tag{2}$$

Let us define the lineality space of K, $K \bigcup (-K)$, by $\ell(K)$. Since the $s \times p$ matrix U_1 has the maximal rank p,

$$\begin{aligned} \ell(K) &= \{(u, y) : U_1 y + U_2 u = 0\} \\ &= \{(u, y) : y + U_1 u = 0\}. \end{aligned}$$

Since row vectors of U_1 are generators of D^o, $MQ \subset D$ implies that every row vector of M_2 is in Q^o.

DEFINITION 3.4. *We say that K supports D-epi W at (u^*, y^*) if $K \cup [cl(D\text{-epi } W) - (u^*, y^*)] \subset \ell(K)$. We call (u^*, y^*) the supporting point of D-epi W. The set $\tilde{K} = K + (u^*, y^*)$ is called a conical variety of K. We also say that $\tilde{K}$ supports D-epi W at (u^*, y^*).*

The existence of a supporting conical variety of D-epi W at a properly efficient point is guaranteed by the following theorem.

THEOREM 3.5. *Assume Slater's constraint qualification on (P). If Ω^* is a properly efficient D-solution to (P), then there exists a supporting conical variety K of D-epi W at $(G(\Omega^*), F(\Omega^*))$.*

Proof. Since Ω^* is properly efficient, from Theorem 2.1, there exists an M^* such that

$$(3) \qquad F(\Omega) + M^* G(\Omega) \not\leq_D F(\Omega^*) + M^* G(\Omega^*) \text{ for all } \Omega \in \mathcal{S}.$$

If $(u, y) \in D$-epi W, then $y \in W(u) + D$ so that for some $\Omega' \in \mathcal{S}(u)$, $F(\Omega') \leqq_D y$ and $G(\Omega\prime) \leqq_Q u$. Let $K = \{(u, y) : U_1 y + U_1 M^* u \leqq 0\}$, where U_1 is given in (1). Then from (3) and $F(\Omega') + M^* G(\Omega) \not\leq_D y + M^* u$, we have

$$y + M^* u \not\leq_D F(\Omega^*) + M^* G(\Omega^*).$$

Therefore, by Lemma 3.2,

$$(4) \qquad U_1(y - F(\Omega^*)) + U_2(u - G(\Omega^*)) \not\leq 0.$$

Since $(u, y) \in D$-epi W was arbitrary, (4) holds for all $(u, y) \in D$-epi W. Then, equivalently, K supports D-epi W at $(G(\Omega^*), F(\Omega^*))$.

We consider a relationship between a conical variety and supporting hyperplanes. First we define several kinds of half-spaces associated with a hyperplane. Let $H(\lambda, \mu : r)$ be a hyperplane in $R^m \times R^p$ with the normal (λ, μ) such that

$$H(\lambda, \mu : r) = \{(u, y) \in R^m \times R^p :< \mu, y > + < \lambda, u > -r = 0\}.$$

Associated with the hyperplane $H(\lambda, \mu : r)$, half-spaces are defined as follows :

$$H^+(\lambda, \mu : r) = \{(u, y) \in R^m \times R^p :< \mu, y > + < \lambda, u > -r \geq 0\}$$

$$H^o_+(\lambda, \mu : r) = \{(u, y) \in R^m \times R^p :< \mu, y > + < \lambda, u > -r > 0\}$$

Similarly, H_- and H^o_- are defined by replacing $\geq$ (respectively , $=$) with $\leq$ (respectively, $=$).

LEMMA 3.6. [7,Lemma 2.6] *The lineality space of the cone K defined in (2) with $U_2 = U_1 M$ is included in the hyperplane $H(\lambda, \mu : 0)$ if and only if the matrix satisfies $M^t \mu = \lambda$.*

Let $H(\lambda, \mu)$ denote the supporting hyperplane [8, pp.99-100] for D-epi W with the inner normal (λ, μ), that is , $H(\lambda, \mu) = H(\lambda, \mu : \tilde{r})$, where $\tilde{r} = sup\{r : H(\lambda, \mu : r) \supset D\text{-epi } W\}$.

LEMMA 3.7. [7, Lemma 2.7] *For any supporting hyperplane $H(\lambda, \mu)$ for D-epi W at some $(u, y) \in D$-epi W, we have that $\lambda \in Q^o$ and $\mu \in D^o$.*

The next two theorems clarify the relationship between supporting hyperplanes and supporting conical varieties. Note that similar results of Nakayama [7] are not applied directly because the D-epi W is not guaranteed to be convex in the programming problem with set functions. Thus we assume more restriction on the perturbed feasible set $\{\Omega \in \mathcal{S} : G(\Omega) \leqq_Q u\}$ in the second theorem.

THEOREM 3.8. *Let $H(\lambda, \mu)$ be the supporting hyperplane for $D - epiW$ with supporting point (u^*, y^*). Assume that μ is in D^o. Let K be a cone defined in (9) for some $M \in \mathcal{M}$, and let $\ell(\tilde{K})$ be the linear* *variety of $\ell(K)$ passing through (u^*, y^*). If $\ell(\tilde{K})$ is included in $H(\lambda, \mu)$, then K supports D-epi W at (u^*, y^*).*

Proof. It is similar to that of [7, Lemma 2.8]. Conversely, given a conical variety of a cone K, we have a supporting hyperplane of $D - epiW$.

THEOREM 3.9. *Assume that $\zeta^o = \zeta$ in the problem (P). If some conical variety $\tilde{K}$ of a cone K supports D-epi W at (u^*, y^*), then there exists a hyperplane $H(\lambda, \mu : r)$ with $\mu \neq 0$ supporting D-epi W at (u^*, y^*) such that*

$$\ell(\tilde{K}) \subset H(\lambda, \mu : r).$$

Proof. Suppose that $K = \{(u, y) \in R^m \times R^p : U_1 + U_1 M u \leqq 0\}$ for some M and that its conical variety K supports D-epi W, at (u^*, y^*). Then $U_1(y - y^*) + U_1 M(u - u^*) \nleq 0$ for all $(u, y) \in D$-epi W, or $y + Mu \nleq_D y^* + Mu^*$ for all $(u, y) \in D$-epi W. Note that

$$\begin{aligned} &\{U_1 y + U_1 M u : (u, y) \in D\text{-epi } W\} \\ &\subset \{U_1 y + U_1 M u : (u, y) \in D\text{-epi } \overline{W}\} \end{aligned} \tag{5}$$

where $\overline{W}(u) = Min_D cl(F(\mathcal{S}(u)))$ and $\mathcal{S}(u) = \{\Omega \in \mathcal{S} : G(\Omega) \leqq_Q u\}$. Since ζ^o is convex and $\overline{W}$ is D-convex, since $\overline{W}$ is a D-convex point -to-set map on the convex set ζ^o, it follows that the set defined in (5) is a convex set. Using Lemma 3.7 for D-epi $\overline{W}$, we obtain the result. The rest of the proof is similar to that of [7, Lemma 2.8].

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