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REMARKS ON SUBMANIFOLDS OF COSYMPLECTIC MANIFOLDS

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1. Introduction.

A normal almost contact structure such that its fundamental 2-form and contact form are both closed is called cosymplectic structure. A Riemannian manifold with a cosymplectic structure is said to be cosymplectic manifold [1, 2, 3]. In this paper, we are to construct almost complex structure F on a product manifold of an anti-invariant submanifold of cosymplectic manifolds and Euclidean spaces. Moreover, we study the necessary and sufficient conditions for F is integrable. In the last, we introduce the almost Hermitian structure on the product manifold. Unless otherwise stated we use in the present paper the system of indices as follows:

$$\begin{aligned} A, B, C, D, \cdots &: 1, 2, \cdot, 2m+2 \\ a, b, c, d, \cdots &: 1, 2, \cdots, n \\ x, y, z, w, \cdots &: n+1, \cdots, n+p = 2m+1 \\ * &: 2m+2. \end{aligned}$$

2. Anti-invariant submanifolds of cosymplectic manifolds.

Let M be a $(2m+1)$-dimensional cosymplectic manifold with structure (ϕ, ξ, η, G), that is, a manifold M which admits a 1-form η, a vector field ξ, a metric tensor G satisfying

$$\begin{aligned} &\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \\ &G(\xi, X) = \eta(X), \quad \eta(\xi) = 1, \\ &G(\phi X, \phi Y) = G(X, Y) - \eta(X)\eta(Y) \end{aligned} \tag{2.1}$$

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$$(2.2) \qquad d\phi = 0, \quad d\xi = 0$$

for any vector fields X, Y on M. The fundamental 2-form Φ is defined by

$$(2.3) \qquad \Phi(X,Y) = G(\phi X, Y).$$

It can be shown in [1] that the cosymplectic structure is characterized by

$$(2.4) \qquad \nabla_X \Phi = 0 \quad and \quad \nabla_X \eta = 0,$$

where ∇ is the connection of G.

Let N be an n-dimensional anti–invariant submanifold of M, that is,

$$\phi(T_x(N)) \subset T_x(M)^\perp$$

for each point x in N, where $T_x(N)$ and $T_x(M)^\perp$ denote the tangent and normal space of N at x, respectively. The induced metrics on the tangent and normal space are respectively defined by

$$g_{cb} = G(B_c, B_b)$$
$$g_{xy} = G(C_x, C_y).$$

We denote the inverse components of $g = (g_{cb})$ and $\bar{g} = (g_{xy})$ by $g^{-1} = (g^{cb})$ and $\bar{g}^{-1} = (g^{xy})$ respectively. Then we get [4,5,6],

$$(2.5) \qquad \begin{aligned} &\phi_x^{\ y}\phi_y^{\ z} = -\delta_x^{\ z} + \xi_x\xi^z + f_x^{\ a}f_a^{\ z},\\ &\phi_x^{\ b}\xi^x = 0,\\ &\phi_a^{\ x}\eta^a = \phi_z^{\ x}\xi^z,\\ &\eta^a\eta_a + \xi_x\xi^x = 1,\\ &\phi_c^{\ x}\phi_x^{\ y} = -\eta_c\xi^y,\\ &\phi_c^{\ x}\phi_x^{\ b} = \delta_c^{\ b} - \eta_c\eta^b, \end{aligned}$$

where we have put

$$\begin{aligned} G(\phi B_c, C_x) &= -\phi_{cx},\\ G(\phi C_x, B_a) &= \phi_{xa},\\ G(\phi C_x, C_y) &= \phi_{xy},\\ G(\xi, B_a) &= \eta_a,\\ G(\xi, C_x) &= \xi_x \end{aligned}$$

for B_c and C_x the local basis of the tangent space and normal space of N, and

$$\begin{aligned}\phi_{xy}g^{yz} &= \phi_x{}^z,\\ \phi_{xa}g^{ab} &= \phi_x{}^b,\\ \eta_a g^{ab} &= \eta^b,\\ \xi_x g^{xy} &= \xi^y.\end{aligned}$$

We easily see that

$$\phi_{xy} = -\phi_{yx} \quad \text{and} \quad \phi_{cx} = \phi_{xc}.$$

Let ∇_d be the induced Riemannian connection on N, then we get [4]

$$\begin{aligned}\nabla_d\phi_x^{\,y} &= h_{da}{}^x\phi_a^{\,y} - h_{da}{}^y f_x{}^a,\\ \nabla_d\phi_c^{\,y} &= -h_{dc}{}^x f_x{}^y,\\ \nabla_d\eta^a &= h_{da}{}^x\xi^x,\\ \nabla_d\xi_x &= -h_{da}{}^x\eta^a,\\ h_{bc}{}^x\phi_x{}^a &= \phi_c{}^x h_{ba}{}^x.\end{aligned} \tag{2.6}$$

3. An almost complex structure on $N \times R^p \times R^1$.

In this section, we consider the product manifold $N \times R^p \times R^1$, R^p being a p–dimensional Euclidean space. Denote $N \times R^p \times R^1$ by $\tilde{M}$ and define on $\tilde{M}$ a tensor field F of type (1.1) with local components F_B^A given by

$$(F_B^A) = \begin{pmatrix} 0 & \phi_b^{\,x} & \eta_b \\ -\phi_y^{\,a} & \phi_y^{\,x} & -\xi_y \\ -\eta^a & \xi^x & 0 \end{pmatrix} \tag{3.1}$$

in $\{\tilde{M}, x^A\}$, $\{N, x^a\}$ being a coordinate neighborhood of N and $(x^{n+1}, \cdots, x^{n+p} = x^{2m+1})$ being a Cartesian coordinate in R^p and $x^* = x^{2m+2}$ natural coordinate in R^1. Then, taking account of (2.5), we can see that $F^2 = -I$ holds on $\tilde{M}$. Thus we have

PROPOSITION 3.1. *If N is an anti-invariant submanifold of cosymplectic manifolds, then $\tilde{M} = N \times R^p \times R^1$ is an almost complex manifold.*

The Nijenhuis tensor $[F,F]_{CB}^{A}$ of the almost complex structure F has local components

$$[F,F]_{CB}^{A} = F_C^{\ E}\partial_E F_B^{\ A} - F_B^{\ E}\partial_E F_C^{\ A} - (\partial_C F_B^{\ E} - \partial_B F_C^{\ E})F_E^{\ A}$$

on $\tilde{M}$. We denote $[F,F]_{CB}^{A}$ by $N_{CB}^{\ A}$. Then, using (3.1), the components of $N_{CB}^{\ A}$ are given by as follows;

$$N_{yz}^{\ x} = \phi_x^{\ e}\partial_e\phi_y^{\ z} - \phi_y^{\ e}\partial_e\phi_x^{\ z}, \tag{3.2}$$

$$N_{yz}^{\ a} = \phi_y^{\ e}\partial_e\phi_z^{\ a} - \phi_z^{\ e}\partial_e\phi_y^{\ a}, \tag{3.3}$$

$$N_{yz}^{\ *} = \phi_y^{\ e}\partial_e\xi_x - \phi_x^{\ e}\partial_e\xi_y \tag{3.4}$$

and so forth.

It is well known that the Nijenhuis tensor of an almost complex structure F satisfies the condition [6]

$$N_{CE}^{\ A}F_B^{\ E} + N_{CB}^{\ E}F_E^{\ A} = 0. \tag{3.5}$$

Substituting (3.1) into (3.5), we have

(3.6-1)

$$N_{cy}{}^{a}\phi_{x}{}^{y} - N_{ce}{}^{a}\phi_{x}{}^{e} - N_{cx}{}^{y}\phi_{y}{}^{a} - N_{c*}{}^{a}\xi_{x} + N_{cx}{}^{*}\eta^{a} = 0,$$

(3.6-2)

$$N_{cx}{}^{e}\phi_{e}{}^{y} - N_{ce}{}^{y}\phi_{x}{}^{e} + N_{cz}{}^{y}\phi_{x}{}^{z} + N_{cx}{}^{z}\phi_{z}{}^{y} - N_{c*}{}^{y}\xi_{x} + N_{cx}{}^{*}\xi^{y} = 0,$$

(3.6-3)

$$N_{cx}{}^{e}\eta_{e} - N_{ce}{}^{*}\phi_{x}{}^{e} + N_{cy}{}^{*}\phi_{x}{}^{y} - N_{cx}{}^{y}\xi_{y} - N_{c*}{}^{*}\xi_{x} = 0,$$

(3.6-4)

$$N_{xz}{}^{a}\phi_{y}{}^{z} - N_{xe}{}^{a}\phi_{y}{}^{e} - N_{xy}{}^{z}\phi_{z}{}^{a} - N_{x*}{}^{a}\xi_{y} - N_{xy}{}^{*}\eta^{a} = 0,$$

(3.6-5)

$$N_{xz}{}^{e}\phi_{e}{}^{y} - N_{xe}{}^{y}\phi_{z}{}^{e} + N_{xw}{}^{y}\phi_{z}{}^{w} + N_{xz}{}^{w}\phi_{w}{}^{y} - N_{x*}{}^{y}\xi_{z} + N_{xz}{}^{*}\xi^{y} = 0,$$

(3.6-6)

$$N_{xy}{}^{e}\eta_{e} - N_{xe}{}^{*}\phi_{y}{}^{e} + N_{xz}{}^{*}\phi_{y}{}^{z} - N_{xy}{}^{z}\xi_{z} - N_{x*}{}^{x}\xi_{y} = 0,$$

(3.6-7)

$$N_{*y}{}^{a}\phi_{x}{}^{y} - N_{*e}{}^{a}\phi_{x}{}^{e} + N_{*x}{}^{y}\phi_{y}{}^{a} - N_{*x}{}^{*}\eta^{a} = 0,$$

(3.6-8)

$$N_{*x}{}^{e}\phi_{e}{}^{y} - N_{*e}{}^{y}\phi_{x}{}^{e} + N_{*z}{}^{y}\phi_{x}{}^{z} + N_{*x}{}^{z}\phi_{z}{}^{y} + N_{*x}{}^{*}\xi_{y} = 0,$$

(3.6-9)

$$N_{*x}{}^{e}\eta_{e} - N_{*e}{}^{*}\phi_{x}{}^{e} + N_{*y}{}^{*}\phi_{x}{}^{y} - N_{*x}{}^{y}\xi_{y} = 0.$$

Now we assume that $N_{yz}{}^{x}, N_{yz}{}^{a}, N_{yz}{}^{*}$ are vanish identically and the function $\lambda^2 = g_{xy}\xi^x\xi^y$ does not vanish almost everywhere. The the equation (3.6-4) becomes

$$N_{xe}{}^{a}\phi_{y}{}^{e} + N_{x*}{}^{a}\xi_{y} = 0. \tag{3.7}$$

Transvecting (3.7) with ξ^y, then we get $N_{x*}{}^{a} = 0$, so that (3.7) is reduced to

$$N_{xe}{}^{a}\phi_{y}{}^{e}0. \tag{3.8}$$

Transvecting $\phi_{c}{}^{y}$ and η^c successively, we get $N_{zc}{}^{a} = 0$. By the same method, we obtain $N_{z*}{}^{x} = 0$ and $N_{zc}{}^{x} = 0$ from (3.6-5), $N_{z*}{}^{*} = 0$ and

$N_{ze}^{\ *} = 0$ from (3.6-6) respectively. Then the equations (3.6-7), (3.6-8) and (3.6-9) are reduced to

$$N_{*e}^{\ a}\phi_y^{\ e} = 0, \tag{3.9}$$

$$N_{*e}^{\ x}\phi_y^{\ e} = 0, \tag{3.10}$$

$$N_{*e}^{\ *}\phi_y^{\ e} = 0. \tag{3.11}$$

Transvecting $\phi_c^{\ y}$ and using (2.5), we get $N_{*e}^{\ a} = 0$, $N_{*e}^{\ x} = 0$ and $N_{*e}^{\ *} = 0$. Henceforth the equations (3.6-1), (3.6-2) and (3.6-3) become

$$N_{ce}^{\ a}\phi_y^{\ e} = 0, \tag{3.12}$$

$$N_{ce}^{\ x}\phi_y^{\ e} = 0, \tag{3.13}$$

$$N_{ce}^{\ *}\phi_y^{\ e} = 0. \tag{3.14}$$

By the same method of the above statements, we obtain $N_{ce}^{\ a} = 0$, $N_{ce}^{\ x} = 0$ and $N_{ce}^{\ *} = 0$. Thus we have

THEOREM 3.2. *Let N be an anti-invariant submanifold of cosymplectic manifolds and let the function λ^2 does not vanish almost everywhere. If the three components $N_{yz}^{\ x}$, $N_{yz}^{\ a}$ and $N_{yz}^{\ *}$ are identically vanish on N, then the other components are all vanish.*

It is well known that [3,6]

THEOREM 3.3. *In order that an almost complex structure F be integrable, it is necessary and sufficient that the components of N are vanish.*

Considering Theorems 3.2 and 3.3, we get

THEOREM 3.4. *Let N be an anti-invariant submanifold of cosymplectic manifolds and let the function λ^2 does not vanish almost everywhere. Then the almost complex structure F is integrable if and only if the components $N_{yz}^{\ x}$, $N_{yz}^{\ a}$ and $N_{yz}^{\ *}$ are identically vanish on N.*

By use of the induced Riemannian connection ∇_d on N, we can calculate the components of $N_{yz}^{\ x}$, $N_{yz}^{\ a}$ and $N_{yz}^{\ *}$ as follows:

$$N_{yz}^{\ x} = (h_e^{\ d}{}_y\phi_z^{\ e} - h_e^{\ d}{}_z\phi_y^{\ e})\phi_d^{\ x}, \tag{3.15}$$

$$N_{yx}^{\ a} = h_d^{\ az}(\phi_{zy}\phi_x^{\ d} - \phi_{zx}\phi_y^{\ d}), \tag{3.16}$$

$$N_{yx}^{\ *} = (h_{dey}\phi_x^{\ e} - h_{dex}\phi_y^{\ e})\eta^d, \tag{3.17}$$

with the aid of (2.6). Assume that $N_{yz}^{x} = 0$ and apply $\phi_x^{\ b}$ and η_b successively to (3.15), we get

$$(3.18) \qquad (h_e{}^d{}_y \phi_z^{\ e} - h_e{}^d{}_z \phi_y^{\ e})\eta_d = 0,$$

that is, $N_{yz}^{*} = 0$. Therefore we can state the Theorem 3.4 as follows.

THEOREM 3.5. *Let N be an anti-invariant submanifold of cosymplectic manifolds and let the function λ^2 does not vanish almost everywhere. Then the almost complex structure F is integrable if and only if the components N_{yz}^{x}, N_{yz}^{a} are identically vanish on N.*

4. Almost Hermitian structure on $\tilde{M}$.

Let N be an anti-invariant submanifold of cosymplectic manifolds. If we consider a Riemannian metric H on $\tilde{M}$ with components

$$(4.1) \qquad H = (h_{CB}) = \begin{pmatrix} g_{cb} & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix},$$

then we can see that

$$(4.2) \qquad H_{CB} F_D^C F_E^B = H_{DE},$$

that is, (F, H) defines an almost Hermitian structure on $\tilde{M}$. Thus we can state

THEOREM 4.1. *The almost complex manifold $(\tilde{M}, F)$ becomes an almost Hermitian manifold with a Hermite metric H.*

We denote the christoffel symbol by $\widetilde{\{B\ ^C A\}}$ and $\{b\ ^c a\}$ formed by $H = (h_{CB})$ and $g = (g_{cb})$ respectively. Then, using (4.1), we find

$$\widetilde{\{b\ ^c a\}} = \{b\ ^c a\}$$

and the other are vanish.

Let $\tilde{\nabla}$ be a covariant differentiation with respect to $\{\widetilde{B\ {}^{C}A}\}$ formed by H. Then the covariant derivatives of F are given by

$$\tilde{\nabla}_c F_b{}^x = \nabla_c \phi_b{}^x, \tag{4.3}$$

$$\tilde{\nabla}_c F_b{}^* = \nabla_c \eta_b, \tag{4.4}$$

$$\tilde{\nabla}_c F_y{}^x = \nabla_c \phi_y{}^x, \tag{4.5}$$

$$\tilde{\nabla}_c F_y{}^* = -\nabla_c \xi_y \tag{4.6}$$

and the others are all vanish. Hence we have

PROPOSITION 4.2. *The almost Hermitian manifold* $\tilde{M}$ *to be a Kaehlerian space with* (F, H) *if and only if all of* $\phi_y{}^x$, $\phi_b{}^x$, ξ^x *and* η_b *are covariantly constant on* N.

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