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THE COMPLETION OF SOME FUZZY METRIC SPACE

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1. INTRODUCTION.

D. Dubois and H. Prade introduced the notions of fuzzy numbers and defined its basic operations [2]. R. Goetschel, W. Voxman, A. Kaufmann, M. Gupta and G. Zhang have done much work about fuzzy numbers [3,4,5,7].

Let $\mathbb{R}$ be the set of all real numbers and $F^*(\mathbb{R})$ all fuzzy subsets defined on $\mathbb{R}$. G. Zhang [7] defined the fuzzy number $\tilde{a} \in F^*(\mathbb{R})$ as follows :

(1) $\tilde{a}$ is normal,

(2) for every $\lambda \in (0, 1]$, $a_\lambda = \{x \mid \tilde{a}(x) \geq \lambda\}$ is a closed interval, denoted by $[a_\lambda^-, a_\lambda^+]$.

Now, let us denote the set of all fuzzy numbers on the real line $\mathbb{R}$ defined by G. Zhang as $F(\mathbb{R})$.

The purpose of this paper is to prove that the fuzzy metric space $(F(\mathbb{R}), \tilde{\rho})$ can be completed by using the equivalence classes of Cauchy sequences, where $\tilde{\rho}$ is defined by

$$\tilde{\rho}(\tilde{a}, \tilde{b}) = \bigcup_{\lambda\in[0,1]} \lambda[|a_1^- - b_1^-|, \sup_{\lambda\leq\eta\leq1} |a_\eta^- - b_\eta^-| \vee |a_\eta^+ - b_\eta^+|].$$

In section 2, we quote basic definitions and theorems from [7,8] which will be needed in the proof of main theorem. In section 3, after defining the fuzzy isometry and the completion concepts, we prove main theorem :

「 The fuzzy metric space $(F(\mathbb{R}), \tilde{\rho})$ has a completion $(\hat{F}(\mathbb{R}), \hat{\tilde{\rho}})$ which has a subspace X that is fuzzy isometric with $F(\mathbb{R})$ and is dense in $\hat{F}(\mathbb{R})$. This space $(\hat{F}(\mathbb{R}), \hat{\tilde{\rho}})$ is uniquely determined up to a fuzzy isometry.」

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2. BASIC DEFINITIONS AND RESULTS.

In this section, we quote basic definitions and results without proof from [7,8] which will be needed in the proof of main theorem.

Let $\mathbb{R}$ be the set of all real numbers and $F^*(\mathbb{R})$ all fuzzy subsets defined on $\mathbb{R}$.

DEFINITION 2.1. Let $\tilde{a} \in F^*(\mathbb{R})$. $\tilde{a}$ is called a fuzzy number if $\tilde{a}$ has the properties :

(1) $\tilde{a}$ is normal, i.e., there exists $x \in \mathbb{R}$ such that $\tilde{a}(x) = 1$,
(2) whenever $\lambda \in (0, 1]$, then $a_\lambda = \{x \mid \tilde{a}(x) \geq \lambda\}$ is a closed interval, denoted by $[a_\lambda^-, a_\lambda^+]$.

Let $F(\mathbb{R})$ be the set of all fuzzy numbers on the real line $\mathbb{R}$.

By the decomposition theorem of fuzzy sets,

$$\tilde{a} = \bigcup_{\lambda \in [0,1]} \lambda[a_\lambda^-, a_\lambda^+]$$

for every $\tilde{a} \in F(\mathbb{R})$. If we define $\tilde{a}(x)$ by

$$\begin{aligned} \tilde{a}(x) &= 1 \quad \text{for } x = k, \\ &= 0 \quad \text{for } x \neq k \quad (k \in \mathbb{R}), \end{aligned}$$

then $k \in F(\mathbb{R})$ and $k = \cup_{\lambda \in [0,1]} \lambda[k, k]$.

DEFINITION 2.2. For every $\tilde{a}, \tilde{b}, \tilde{c} \in F(\mathbb{R})$, we say that $\tilde{c} = \tilde{a} + \tilde{b}$ if for every $\lambda \in (0, 1]$, $c_\lambda^- = a_\lambda^- + b_\lambda^-$ and $c_\lambda^+ = a_\lambda^+ + b_\lambda^+$. We say that $\tilde{c} = \tilde{a} - \tilde{b}$ if for every $\lambda \in (0, 1]$, $c_\lambda^- = a_\lambda^- - b_\lambda^+$ and $c_\lambda^+ = a_\lambda^+ - b_\lambda^-$.

DEFINITION 2.3. For every $k \in \mathbb{R}$ and every $\tilde{a} \in F(\mathbb{R})$, we define

$$\begin{aligned} k\tilde{a} &= \bigcup_{\lambda \in [0,1]} \lambda[ka_\lambda^-, ka_\lambda^+] \quad \text{if } k \geq 0, \\ &= \bigcup_{\lambda \in [0,1]} \lambda[ka_\lambda^+, ka_\lambda^-] \quad \text{if } k < 0. \end{aligned}$$

Note that we can find in [6] the definitions of multiplication, division, maximal and minimal operations of the fuzzy numbers.

DEFINITION 2.4. For $\tilde{a}, \tilde{b} \in F(\mathbb{R})$, we say that $\tilde{a} \leq \tilde{b}$ if for every $\lambda \in (0, 1]$, $a_\lambda^- \leq b_\lambda^-$ and $a_\lambda^+ \leq b_\lambda^+$. We say that $\tilde{a} < \tilde{b}$ if $\tilde{a} \leq \tilde{b}$ and there exists $\lambda \in (0, 1]$ such that $a_\lambda^- < b_\lambda^-$ or $a_\lambda^+ < b_\lambda^+$. We say that $\tilde{a} = \tilde{b}$ if $\tilde{a} \leq \tilde{b}$ and $\tilde{b} \leq \tilde{a}$.

DEFINITION 2.5. A fuzzy distance of fuzzy numbers is a function $\tilde{\rho} : F(\mathbb{R}) \times F(\mathbb{R}) \to F(\mathbb{R})$ with the properties :

(1) $\tilde{\rho}(\tilde{a}, \tilde{b}) \geq 0$, $\tilde{\rho}(\tilde{a}, \tilde{b}) = 0$ iff $\tilde{a} = \tilde{b}$,
(2) $\tilde{\rho}(\tilde{a}, \tilde{b}) = \tilde{\rho}(\tilde{b}, \tilde{a})$,
(3) whenever $\tilde{c} \in F(\mathbb{R})$, we have $\tilde{\rho}(\tilde{a}, \tilde{b}) \leq \tilde{\rho}(\tilde{a}, \tilde{c}) + \tilde{\rho}(\tilde{c}, \tilde{b})$.

If $\tilde{\rho}$ is the fuzzy distance of fuzzy numbers, we call $(F(\mathbb{R}), \tilde{\rho})$ a fuzzy metric space.

We define

$$\tilde{\rho}(\tilde{a}, \tilde{b}) = \bigcup_{\lambda \in [0,1]} \lambda [\, |a_1^- - b_1^-|, \sup_{\lambda \leq \eta \leq 1} |a_\eta^- - b_\eta^-| \vee |a_\eta^+ - b_\eta^+| \,], \qquad (\star)$$

for any $\tilde{a}, \tilde{b} \in F(\mathbb{R})$, where $\vee$ means max.

THEOREM 2.1. *[8] $\tilde{\rho}(\tilde{a}, \tilde{b})$ defined by the equality $(\star)$ is a fuzzy distance of fuzzy numbers.*

THEOREM 2.2. *Whenever $\tilde{a}, \tilde{b}, \tilde{c} \in F(\mathbb{R})$, $k \in \mathbb{R}$, we have*

(1) $\tilde{\rho}(\tilde{a} + \tilde{b},\ \tilde{a} + \tilde{c}) = \tilde{\rho}(\tilde{b},\ \tilde{c})$,
(2) $\tilde{\rho}(\tilde{a} - \tilde{b},\ \tilde{a} - \tilde{c}) = \tilde{\rho}(\tilde{b},\ \tilde{c})$,
(3) $\tilde{\rho}(\tilde{b} - \tilde{a},\ \tilde{c} - \tilde{a}) = \tilde{\rho}(\tilde{b},\ \tilde{c})$,
(4) $\tilde{\rho}(k\tilde{a},\ k\tilde{b}) = |k|\ \tilde{\rho}(\tilde{a},\ \tilde{b})$.

DEFINITION 2.6. Let $\{\tilde{a}_n\} \subset F(\mathbb{R})$, $\tilde{a} \in F(\mathbb{R})$. Then, $\{\tilde{a}_n\}$ is said to converge to $\tilde{a}$ in fuzzy distance $\tilde{\rho}$, denoted by

$$\lim_{n \to \infty} \tilde{a}_n = \tilde{a} \quad \text{or} \quad \tilde{a}_n \to \tilde{a} \text{ as } n \to \infty,$$

if for arbitrary given $\varepsilon > 0$ there exists an integer $N > 0$ such that $\tilde{\rho}(\tilde{a}_n, \tilde{a}) < \varepsilon$ for $n \geq N$.

THEOREM 2.3. *[7] Let* $\{\tilde{a}_n\}, \{\tilde{b}_n\} \subset F(\mathbb{R})$, $\tilde{a}, \tilde{b} \in F(\mathbb{R})$, $k \in \mathbb{R}$. *If*

$$\lim_{n\to\infty} \tilde{a}_n = \tilde{a} \quad \textit{and} \quad \lim_{n\to\infty} \tilde{b}_n = \tilde{b},$$

then

(1) $\lim_{n\to\infty} (\tilde{a}_n \pm \tilde{b}_n) = \tilde{a} \pm \tilde{b}$ *(the same order of sign),*
(2) $\lim_{n\to\infty} k\tilde{a}_n = k\tilde{a}$.

DEFINITION 2.7. Let $\{\tilde{a}_n\} \subset F(\mathbb{R})$. Then $\{\tilde{a}_n\}$ is called a Cauchy sequence, if for any $\varepsilon > 0$ there exists an integer $N > 0$ such that $\tilde{\rho}(\tilde{a}_n, \tilde{a}_m) < \varepsilon$ for $n, m > N$.

DEFINITION 2.8. If a fuzzy metric space has the property that every Cauchy sequence converges, the space is called a complete fuzzy metric space.

THEOREM 2.4. *(Cauchy criterion for convergence). Let* $\{\tilde{a}_n\} \subset F(\mathbb{R})$. *Then* $\{\tilde{a}_n\}$ *is convergent in fuzzy distance* $\tilde{\rho}$ *if and only if* $\{\tilde{a}_n\}$ *is a Cauchy sequence.*

3. Main theorem.

In this section, we prove that the fuzzy metric space $(F(\mathbb{R}), \tilde{\rho})$ has a completion $(\hat{F}(\mathbb{R}), \hat{\tilde{\rho}})$.

DEFINITION 3.1. Let $X_1 = (X_1, \tilde{d}_1)$, $X_2 = (X_2, \tilde{d}_2)$ be fuzzy metric spaces. Then,

(1) A mapping f of X_1 into X_2 is said to be fuzzy isometric or a fuzzy isometry if f preserves fuzzy distances, that is, if for all $\tilde{x}, \tilde{y} \in X_1$, $\tilde{d}_2(f(\tilde{x}), f(\tilde{y})) = \tilde{d}_1(\tilde{x}, \tilde{y})$, where $f(\tilde{x})$ and $f(\tilde{y})$ are the images of $\tilde{x}$ and $\tilde{y}$ respectively.
(2) The space X_1 is said to be fuzzy isometric with the space X_2 if there exists a bijective fuzzy isometry of X_1 onto X_2. The spaces X_1 and X_2 are then called fuzzy isometric spaces.

DEFINITION 3.2. The complete fuzzy metric space $(\hat{X}_1, \hat{\tilde{d}}_1)$ is said to be a completion of the given fuzzy metric space $(X_1, \tilde{d}_1)$ if

(1) $(X_1, \tilde{d}_1)$ is fuzzy isometric with a subspace $(X, \hat{\tilde{d}}_1)$ of $(\hat{X}_1, \hat{\tilde{d}}_1)$,
(2) X is dense in $\hat{X}_1$, i.e., the closure of X, $\overline{X} = \hat{X}_1$.

MAIN THEOREM. *The fuzzy metric space* $(F(\mathbb{R}), \tilde{\rho})$ *has a completion* $(\hat{F}(\mathbb{R}), \hat{\tilde{\rho}})$ *which has a subspace* X *that is fuzzy isometric with* $F(\mathbb{R})$ *and is dense in* $\hat{F}(\mathbb{R})$. *This space* $(\hat{F}(\mathbb{R}), \hat{\tilde{\rho}})$ *is uniquely determined up to a fuzzy isometry, that is, if* $(\check{F}(\mathbb{R}), \check{\tilde{\rho}})$ *is another completion having a dense subspace* Y *fuzzy isometric with* $F(\mathbb{R})$, *then* $\hat{F}(\mathbb{R})$ *and* $\check{F}(\mathbb{R})$ *are fuzzy isometric.*

Proof. The proof is somewhat lengthy. We divide it into four steps (a) to (d). We construct :

(a) $\hat{F}(\mathbb{R}) = (\hat{F}(\mathbb{R}), \hat{\tilde{\rho}})$,

(b) a fuzzy isometry f of $F(\mathbb{R})$ onto X, where $\overline{X} = \hat{F}(\mathbb{R})$.

Then we prove :

(c) completeness of $\hat{F}(\mathbb{R})$,

(d) uniqueness of $\hat{F}(\mathbb{R})$ except for fuzzy isometries.

(a). Construction of $\hat{F}(\mathbb{R}) = (\hat{F}(\mathbb{R}), \hat{\tilde{\rho}})$.

Let $\{\tilde{x}_n\}$ and $\{\tilde{x}'_n\}$ be Cauchy sequences in $F(\mathbb{R})$. Define $\{\tilde{x}_n\}$ to be equivalent to $\{\tilde{x}'_n\}$ written $\{\tilde{x}_n\} \sim \{\tilde{x}'_n\}$, if

$$\lim_{n\to\infty} \tilde{\rho}(\tilde{x}_n, \tilde{x}'_n) = 0. \tag{1}$$

Let $\hat{F}(\mathbb{R})$ be the set of all equivalence classes $\hat{x}, \hat{y}, \cdots$ of Cauchy sequences thus obtained. We write $\{\tilde{x}_n\} \in \hat{x}$ to mean that $\{\tilde{x}_n\}$ is a member of $\hat{x}$ (a representative of the class $\hat{x}$). We now set

$$\hat{\tilde{\rho}}(\hat{x}, \hat{y}) = \lim_{n\to\infty} \tilde{\rho}(\tilde{x}_n, \tilde{y}_n) \tag{2}$$

where $\{\tilde{x}_n\} \in \hat{x}$ and $\{\tilde{y}_n\} \in \hat{y}$. We show that this limit exists. By the triangle inequality, we have

$$\tilde{\rho}(\tilde{x}_n, \tilde{y}_n) \leq \tilde{\rho}(\tilde{x}_n, \tilde{x}_m) + \tilde{\rho}(\tilde{x}_m, \tilde{y}_m) + \tilde{\rho}(\tilde{y}_m, \tilde{y}_n),$$
$$\tilde{\rho}(\tilde{x}_m, \tilde{y}_m) \leq \tilde{\rho}(\tilde{x}_n, \tilde{x}_m) + \tilde{\rho}(\tilde{x}_n, \tilde{y}_n) + \tilde{\rho}(\tilde{y}_m, \tilde{y}_n).$$

Hence, because $\tilde{\rho}(\tilde{x}_n, \tilde{y}_n)$, $\tilde{\rho}(\tilde{x}_m, \tilde{y}_m)$ are fuzzy numbers, we obtain

$$\tilde{\rho}\big(\tilde{\rho}(\tilde{x}_n, \tilde{y}_n), \tilde{\rho}(\tilde{x}_m, \tilde{y}_m)\big) \leq \tilde{\rho}(\tilde{x}_n, \tilde{x}_m) + \tilde{\rho}(\tilde{y}_m, \tilde{y}_n). \tag{3}$$

Since $\{\tilde{x}_n\}$ and $\{\tilde{y}_n\}$ are Cauchy sequences, we can make the right side as small as we please. This implies that the limit in (2) exists because $(F(\mathbb{R}), \tilde{\rho})$ is complete.

We must also show that the limit in (2) is independent of the particular choice of representatives. In fact, if $\{\tilde{x}_n\} \sim \{\tilde{x}'_n\}$ and $\{\tilde{y}_n\} \sim \{\tilde{y}'_n\}$, then by (1), (3),

$$\tilde{\rho}\big(\tilde{\rho}(\tilde{x}_n, \tilde{y}_n), \tilde{\rho}(\tilde{x}'_n, \tilde{y}'_n)\big) \le \tilde{\rho}(\tilde{x}_n, \tilde{x}'_n) + \tilde{\rho}(\tilde{y}_n, \tilde{y}'_n) \to 0$$

as $n \to \infty$, which implies the assertion

$$\lim_{n\to\infty} \tilde{\rho}(\tilde{x}_n, \tilde{y}_n) = \lim_{n\to\infty} \tilde{\rho}(\tilde{x}'_n, \tilde{y}'_n).$$

We prove that $\hat{\tilde{\rho}}$ in (2) is a metric on $\hat{F}(\mathbb{R})$. Obviously, $\hat{\tilde{\rho}}$ satisfies $\hat{\tilde{\rho}}(\hat{x}, \hat{y}) \ge 0$ (see Definition of $\tilde{\rho}(\tilde{a}, \tilde{a})$) as well as $\hat{\tilde{\rho}}(\hat{x}, \hat{x}) = 0$ and $\hat{\tilde{\rho}}(\hat{x}, \hat{y}) = \hat{\tilde{\rho}}(\hat{y}, \hat{x})$. Furthermore,

$$\hat{\tilde{\rho}}(\hat{x}, \hat{y}) = 0 \quad \Rightarrow \quad \{\tilde{x}_n\} \sim \{\tilde{y}_n\} \quad \Rightarrow \quad \hat{x} = \hat{y}$$

gives $\hat{\tilde{\rho}}(\hat{x}, \hat{y}) = 0 \Leftrightarrow \hat{x} = \hat{y}$, and the triangle inequality for $\hat{\tilde{\rho}}$ follows from

$$\tilde{\rho}(\tilde{x}_n, \tilde{y}_n) \le \tilde{\rho}(\tilde{x}_n, \tilde{z}_n) + \tilde{\rho}(\tilde{z}_n, \tilde{y}_n)$$

by letting $n \to \infty$.

(b). Construction of a fuzzy isometry $f : F(\mathbb{R}) \to X \subset \hat{F}(\mathbb{R})$.

With each $\tilde{a} \in F(\mathbb{R})$ we associate the class $\hat{a} \in \hat{F}(\mathbb{R})$ which contains the constant Cauchy sequence $\{\tilde{a}, \tilde{a}, \cdots\}$. This defines a mapping $f :$ $F(\mathbb{R}) \to X$ onto the subspace $X = f(F(\mathbb{R})) \subset \hat{F}(\mathbb{R})$. The mapping f is given by $\tilde{a} \mapsto \hat{a} = f(\tilde{a})$, where $\{\tilde{a}, \tilde{a}, \cdots\} \in \hat{a}$. We see that f is a fuzzy isometry since (2) becomes simply

$$\hat{\tilde{\rho}}(\hat{a}, \hat{b}) = \tilde{\rho}(\tilde{a}, \tilde{b}),$$

here $\hat{b}$ is the class of $\{\tilde{y}_n\}$ where $\tilde{y}_n = \tilde{b}$ for all n. Any fuzzy isomerty is injective, and $f : F(\mathbb{R}) \to X$ is surjective since $f(F(\mathbb{R})) = X$. Hence X and $F(\mathbb{R})$ are fuzzy isometric.

We show that X is dense in $\hat{F}(\mathbb{R})$. We consider any $\hat{x} \in \hat{F}(\mathbb{R})$. Let $\{\tilde{x}_n\} \in \hat{x}$. For every $\varepsilon > 0$ there is an integer $N > 0$ such that

$$\tilde{\rho}(\tilde{x}_n, \tilde{x}_N) < \varepsilon/2 \ \text{ for } n > N.$$

Let $\{\tilde{x}_N, \tilde{x}_N, \cdots\} \in \hat{x}_N$. Then $\hat{x}_N \in X$. By (2),

$$\hat{\tilde{\rho}}(\hat{x}, \hat{x}_N) = \lim_{n\to\infty} \tilde{\rho}(\tilde{x}_n, \tilde{x}_N) \le \varepsilon/2 < \varepsilon.$$

This shows that every ε-neighborhood of the arbitrary $\hat{x} \in \hat{F}(\mathbb{R})$ cotains an element of X. Hence X is dense in $\hat{F}(\mathbb{R})$.

(c). Completeness of $\hat{F}(\mathbb{R})$.

Let $\{\hat{x}_n\}$ be any Cauchy sequence in $\hat{F}(\mathbb{R})$. Since X is dense in $\hat{F}(\mathbb{R})$, for every $\hat{x}_n \in \hat{F}(\mathbb{R})$ there is a $\hat{z}_n \in X$ such that

$$\hat{\tilde{\rho}}(\hat{x}_n, \hat{z}_n) < \frac{1}{n}. \tag{4}$$

Hence, by the triangle inequality,

$$\begin{aligned}\hat{\tilde{\rho}}(\hat{z}_m, \hat{z}_n) &\le \hat{\tilde{\rho}}(\hat{z}_m, \hat{x}_m) + \hat{\tilde{\rho}}(\hat{x}_m, \hat{x}_n) + \hat{\tilde{\rho}}(\hat{x}_n, \hat{z}_n) \\ &< \frac{1}{m} + \hat{\tilde{\rho}}(\hat{x}_m, \hat{x}_n) + \frac{1}{n}\end{aligned}$$

and this is less than any given $\varepsilon > 0$ for sufficiently large m and n because $\{\hat{x}_n\}$ is a Cauchy sequence. Hence $\{\hat{z}_m\}$ is Cauchy in X. Since $f : F(\mathbb{R}) \to X$ is fuzzy isometric and $\hat{z}_m \in X$, the sequence $\{\tilde{z}_m\}$, where $\tilde{z}_m = f^{-1}(\hat{z}_m)$, is Cauchy in $F(\mathbb{R})$. Let $\hat{x} \in \hat{F}(\mathbb{R})$ be the class to which $\{\tilde{z}_m\}$ belongs. We show that $\hat{x}$ is the limit of $\{\hat{x}_n\}$. By (4),

$$\begin{aligned}\hat{\tilde{\rho}}(\hat{x}_n, \hat{x}) &\le \hat{\tilde{\rho}}(\hat{x}_n, \hat{z}_n) + \hat{\tilde{\rho}}(\hat{z}_n, \hat{x}) \\ &< \frac{1}{n} + \hat{\tilde{\rho}}(\hat{z}_n, \hat{x}).\end{aligned} \tag{5}$$

Since $\{\tilde{z}_m\} \in \hat{x} \in \hat{F}(\mathbb{R})$ and $\hat{z}_n \in X$, so that $\{\tilde{z}_n, \tilde{z}_n, \cdots\} \in \hat{z}_n$, the inequality (5) becomes

$$\hat{\tilde{\rho}}(\hat{x}_n, \hat{x}) < \frac{1}{n} + \lim_{m\to\infty} \tilde{\rho}(\tilde{z}_n, \tilde{z}_m)$$

and the right side is smaller than any given $\varepsilon > 0$ for sufficiently large n. Hence the arbitrary Cauchy sequence $\{\hat{x}_n\}$ in $\hat{F}(\mathbb{R})$ has the limit $\hat{x} \in \hat{F}(\mathbb{R})$, and $\hat{F}(\mathbb{R})$ is complete.

(d). Uniqueness of $\hat{F}(\mathbb{R})$ except for isometries.

If $(\check{F}(\mathbb{R}), \check{\tilde{\rho}})$ is another completion with a subspace Y dense in $\check{F}(\mathbb{R})$ and fuzzy isometric with $F(\mathbb{R})$, then for any $\check{x}, \check{y} \in \check{F}(\mathbb{R})$ we have sequences $\{\check{x}_n\}$, $\{\check{y}_n\}$ in Y such that $\check{x}_n \to \check{x}$ and $\check{y}_n \to \check{y}$. By the triangle inequality, we have

$$\check{\tilde{\rho}}(\check{x}, \check{y}) \leq \check{\tilde{\rho}}(\check{x}, \check{x}_n) + \tilde{\rho}(\tilde{x}_n, \tilde{y}_n) + \check{\tilde{\rho}}(\check{y}_n, \check{y})$$

for every n, where $\{\tilde{x}_n, \tilde{x}_n, \cdots\} \in \check{x}_n$ and $\{\tilde{y}_n, \tilde{y}_n, \cdots\} \in \check{y}_n$. Since it is true for every n, it is true in the limit as n becomes infinite, which yields

$$\check{\tilde{\rho}}(\check{x}, \check{y}) \leq \lim_{n \to \infty} \tilde{\rho}(\tilde{x}_n, \tilde{y}_n).$$

But
$$\tilde{\rho}(\tilde{x}_n, \tilde{y}_n) \leq \check{\tilde{\rho}}(\check{x}_n, \check{x}) + \check{\tilde{\rho}}(\check{x}, \check{y}) + \check{\tilde{\rho}}(\check{y}, \check{y}_n)$$

which yields the reverse inequality. Hence

$$\check{\tilde{\rho}}(\check{x}, \check{y}) = \lim_{n \to \infty} \tilde{\rho}(\tilde{x}_n, \tilde{y}_n).$$

In a completly analogous manner, we can also show that

$$\hat{\tilde{\rho}}(\hat{x}, \hat{y}) = \lim_{n \to \infty} \tilde{\rho}(\tilde{x}_n, \tilde{y}_n).$$

Consequently,
$$\hat{\tilde{\rho}}(\hat{x}, \hat{y}) = \check{\tilde{\rho}}(\check{x}, \check{y}),$$

that is, the distance on $\check{F}(\mathbb{R})$ and $\hat{F}(\mathbb{R})$ must be the same. Hence $\check{F}(\mathbb{R})$ and $\hat{F}(\mathbb{R})$ are isometric. $\square$

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