Kangweon-Kyungki Math. Jour. 3 (1995), No. 2, pp. 179–186

STRONG SOLUTIONS FOR NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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1. Introduction

Let X be a real Banach space with norm $\|\cdot\|$. We let C denote the space of all continuous functions $f:[-r,0]\to X$ for fixed $r>0$. We consider the abstract functional differential equations of the type

$$(FDE)\qquad \begin{aligned} &x'(t)+A(t,x_t)x(t)\ni G(t,x_t), \quad t\in[0,T],\\ &x_0=\phi, \quad -r\le t\le 0\end{aligned}$$

in a general Banach space, where for $f:[-r,T]\to X$, $f_t(s)=f(t+s)$, $t\in[0,T]$, $s\in[-r,0]$ with a positive constant T. The following conditions will be used in the sequel.

(A.1) For each $(t,\psi)\in[0,T]\times C$, $A(t,\psi):D(A(t,\psi))\subset X\to 2^X$ is m-accretive in X, where $D(A(t,\psi))$ is only dependent on t. We denote $D(A(t,\psi))=D(t)$.

(A.2) For each $t,s\in[0,T]$, $\psi_1,\psi_2\in C$,$v\in X$,

$$\|A_\lambda(t,\psi_1)v-A_\lambda(s,\psi_2)v\|\le L_1(\|v\|)[|t-s|(1+\|A_\lambda(s,\psi_2)v\|)+\|\psi_1-\psi_2\|_C]$$

where $L_1:\mathcal{R}^+\to\mathcal{R}^+=[0,\infty)$ is an increasing, continuous function.

(A.3) For $t,s\in[0,T]$, and $\psi,\psi_1,\psi_2\in C$,

$$\begin{aligned}\|G(t,\psi_1)-G(t,\psi_2)\|&\le\beta\|\psi_1-\psi_2\|_C,\\ \|G(t,\psi)-G(s,\psi)\|&\le L_2(\|\psi\|_C)|t-s|,\end{aligned}$$

where β is a positive constant and $L_2:\mathcal{R}^+\to\mathcal{R}^+$ is a nondecreasing, continuous function.

(A.4) ϕ is a given Lipschitz continuous function with Lipschitz constant L_0 on $[-r,0]$.

Received May 23, 1995.

An operator $A; D \subset X \to 2^X$ is called accretive if

$$\|x_1 - x_2\| \leq \|x_1 - x_2 + \lambda(y_1 - y_2)\|$$

for every $\lambda > 0$ and every $[x_1, y_1], [x_2, y_2] \in A$. It is called m-accretive if it is accretive and $R(I + \lambda A) = X$ for all $\lambda > 0$. If A is m-accretive, we set

$$|Ax| = \lim_{\lambda \downarrow 0} \|A_\lambda x\|, \quad x \in X,$$

where $A_\lambda = (I - J_\lambda)/\lambda$ with $J_\lambda = (I + \lambda A)^{-1}$. We also set

$$\hat{D} = \{x \in X \, : \, |Ax| < \infty\}.$$

It is known that $D(A) \subset \hat{D}(A) \subset \overline{D(A)}$. For other properties of these operators, the reader is referred to Barbu [1], Crandall and Pazy [2], and Evans [3]. The mapping $F : X \to 2^{X^*}$ is defined by

$$F(x) = \{x^* \in X^* \,|\, (x, x^*) = \|x\|^2 = \|x^*\|^2\}, \, x \in X,$$

where X^* is the dual space of X and the norm on X^* is denoted by $\| \cdot \|$.We recall that for $x, y \in X$,

$$[x, y]_+ = \lim_{t \to 0+} \frac{\|x + ty\| - \|x\|}{t},$$

$$[x, y]_- = \lim_{t \to 0-} \frac{\|x + ty\| - \|x\|}{t},$$

$$< x, y >_s = \sup\{(y, x^*) \,|\, x^* \in F(x)\},$$

$$< x, y >_i = \inf\{(y, x^*) \,|\, x^* \in F(x)\}.$$

The following properties are obvious:

$$< x, y >_s = \|x\| [x, y]_+, < x, y >_i = \|x\| [x, y]_-,$$
$$< x, y >_s = - < x, -y >_i = - < -x, y >_i, < x, y >_i \leq < x, y >_s \leq \|x\| \|y\|.$$

For other properties of $[\cdot, \cdot]_+$ and $[\cdot, \cdot]_-$, we refer the reader to the Lakshimikantham and Leela [10] and Kobayashi [9].

By the virtue of (A.1) and (A.2),it is known that $\hat{D}(A(t,\psi))$ is independent of $(t,\psi) \in [0,T] \times C$. (See Evans [3].) We denote by $\hat{D} \equiv \hat{D}(A(t,\psi))$. By using the fixed point theory and the result of Crandall and Pazy[2], Kartsatos and Parrott [6] and Kartsatos[5] have studied the existence of the generalized solution for the functional differential equation with no perturbation term in (FDE). Recently, Ha,Shin and Jin [4] have established the unique local generalized solution of nonlinear functional integro differential equations in general Banach spaces by employing the Banach contraction principle and the method of lines. In this paper, we obtain the existence and the uniqueness of the strong solution of (FDE) by the relationship between the limit solution and Benilan type's integral solution.

2. Main result

As in Ha, Shin and Jin [4],

THEOREM 1. *Let the hypotheses* $(A.1) - -(A.4)$ *and* $\phi(0) \in \hat{D}$. *Then we can take* $T_1 \in (0,T]$ *such that for any partition* $\{t_j^n\}_{j=0}^n$ *of the interval* $[0,T_1]$ *with* $t_j^n = jh_n = j\frac{T_1}{n}$ *and* $j = 0,1,\cdots,n$, *there exists* $\{w_j^n\}_{j=0}^n$ *which satisfies*

$$(2.1)\quad \frac{w_j^n - w_{j-1}^n}{h_n} + A(t_j^n,(\bar{w}_j^n)_{t_j^n})w_j^n \ni G(t_j^n,(\bar{w}_j^n)_{t_j^n}), j = 1,2.\cdots,n.$$

where

$$\bar{w}_j^n(t) = \begin{cases} \phi(0), & t \in [-r,0] \\ w_k^n, & t \in (t_{k-1}^n, t_k^n] \quad \text{for} \quad k = 1,2,\cdots,j-1 \\ w_j^n, & t \in (t_{j-1}^n, T_1]. \end{cases}$$

The function $z_n(t) = \bar{w}_n^n(t)$ *is said to be an approximate solution of* (FDE).

THEOREM 2. *There exists a uniform limit* $z(t)$ *of* $z_n(t)$ *in Theorem 1 for* (FDE) *which is called a limit solution of* (FDE) *on* $[0,T_1]$. *Moreover, such* $z(t)$ *is Lipschitz continuous on* $[0,T_1]$.

The next theorem shows the existence of the Benilan type's integral solution.

THEOREM 3. *Let $z(t)$ be the limit solution for (FDE) on $[0, T_1]$. Then $z(t)$ is an integral solution of (FDE) on $[0, T_1]$, i.e. $z(t) = \phi(t)$ for $t \in [-r, 0]$, $z(t)$ is continuous on $[0, T_1]$ and satisfies the inequality*

$$\begin{aligned} &\|z(t) - x\| - \|z(\tau) - x\| \\ (2.2) \qquad &\leq \int_\tau^t \{[z(\eta) - x, G(\eta, z_\eta) - y]_+ + C|\eta - \bar{r}|\} d\eta, \end{aligned}$$

for every $[x, y] \in A(\bar{r}, \bar{z}_{\bar{r}}), \bar{r} \in [0, T_1], 0 \leq \tau \leq t \leq T_1$ and for some nonnegative constant C.

Proof. If $z_n(t)(= \bar{w}_n^n(t))$ is an approximate solution of (FDE) with $\lim_{n\to\infty} z_n(t) = z(t)$, then there exists $[w_j^n, \tilde{w}_j^n] \in A(t_j^n, z_{t_j^n})$ such that $j = 1, \cdots, n$

$$(w_j^n - x) - (w_{j-1}^n - x) = h_n(G(t_j^n, (\bar{w}_j^n)_{t_j^n}) - \tilde{w}_j^n).$$

Then

$$\begin{aligned} &\|w_j^n - x\| - \|w_{j-1}^n - x\| \\ &\leq [w_j^n - x, w_j^n - x]_- - [w_j^n - x, w_{j-1}^n - x]_+ \\ (*) \qquad &\leq h_n[w_j^n - x, G(t_j^n, (\bar{w}_j^n)_{t_j^n}) - \tilde{w}_j^n]_- \\ &\leq h_n[w_j^n - x, y - \tilde{w}_j^n]_- + h_n[w_j^n - x, G(t_j^n, (\bar{w}_j^n)_{t_j^n}) - y]_+ \end{aligned}$$

for $y \in A(\bar{r}, z_{\bar{r}})x, \bar{r} \in [0, T_1]$. We observe that

$$\begin{aligned} &\|w_j^n - x\| \\ &\quad \leq \|J_\lambda(t_j^n, z_{t_j^n})(w_j^n + \lambda\tilde{w}_j^n) - J_\lambda(t_j^n, z_{t_j^n})(x + \lambda y)\| \\ &\qquad + \|J_\lambda(t_j^n, z_{t_j^n})(x + \lambda y) - J_\lambda(\bar{r}, z_{\bar{r}})(x + \lambda y)\| \\ &\quad \leq \|(w_j^n + \lambda\tilde{w}_j^n) - (x + \lambda y)\| \\ &\qquad + \lambda\|A_\lambda(t_j^n, z_{t_j^n})(x + \lambda y) - A_\lambda(\bar{r}, z_{\bar{r}})(x + \lambda y)\|, \end{aligned}$$

for each $\lambda > 0$. From $(A.2)$,

$$\begin{aligned} &\frac{\|w_j^n - x\| - \|(w_j^n + \lambda\tilde{w}_j^n) - (x + \lambda y)\|}{\lambda} \\ &\leq L_1(\|x + \lambda y\|)|t_j^n - \bar{r}|(1 + \|y\| + M), \end{aligned}$$

where M is the Lipschitz constant for $z(t)$ on $[0, T_1]$. As $\lambda \to 0+$ in the above inequality,

$$[w_j^n - x, y - \tilde{w}_j^n]_- \le L_1(\|x\|)|t_j^n - \bar{r}|(1 + \|y\| + M).$$

Combining this with $(*)$,

$$\|w_j^n - x\| - \|w_{j-1}^n - x\|$$
$$\le h_n L_1(\|x\|)|t_j^n - \bar{r}|(1 + \|y\| + M) + h_n[w_j^n - x, G(t_j^n, (\bar{w}_j^n)_{t_j^n}) - y]_+.$$

Iterating this for $j = k+1, \cdots, p(k+1 < p)$, we have

$$\begin{aligned} &\|w_p^n - x\| - \|w_k^n - x\| \\ &\le \sum_{i=k+1}^{p} h_n\{[w_i^n - x, G(t_i^n, (\bar{w}_i^n)_{t_i^n}) - y]_+ \\ &+ L_1(\|x\|)|t_i^n - \bar{r}|(1 + \|y\| + M)\} \\ &= \sum_{i=k+1}^{p} h_n\{[w_i^n - x, G(t_i^n, (\bar{w}_i^n)_{t_i^n}) - y]_+ + C|t_i^n - \bar{r}|\} \end{aligned} \tag{2.3}$$

for some constant C. Now, by letting $\tau \in (t_{j-1}^n, t_j^n]$ and $t \in (t_{p-1}^n, t_p^n]$ and set $a_n(\sigma) = t_j^n$ for $\sigma \in (t_{j-1}^n, t_j^n]$. According to the definition of z_n, (2.3) becomes

(2.4)
$$\|z_n(t) - x\| - \|z_n(\tau) - x\|$$
$$\le \int_{t_j^n}^{t_p^n} \{[z_n(a_n(\sigma)) - x, G(a_n(\sigma), (z_n)_{a_n(\sigma)}) - y]_+ + C|a_n(\sigma) - \bar{r}|\}d\sigma.$$

Clearly,$z_n(a_n(\sigma)) \to z_(\sigma)$ as $n \to \infty$, uniformly with respect to $\sigma \in [0, T_1]$. Passing to the limit for $n \to \infty$ in (2.4), we have the desired result. □

DEFINITION. A strong solution of (FDE) on $[0, T_1]$ is a function $z(t)$ which is Lipschitz continuous on $[0, T_1]$, differentiable a.e. on $[0, T_1]$, and satisfies (FDE).

Modifying Ha,Shin and Jin [4], we have the uniqueness of the limit solution of (FDE) from Theorem 3 and the uniqueness of integral solution of (FDE).

THEOREM 4. *The limit solution of (FDE) is a strong solution in a reflexive Banach space X.*

Proof. By the virtue of Theorem 1 and Theorem 2, there exists a limit solution $z(t)$ and such $z(t)$ is Lipschitz continuous on $[0, T_1]$. Since X is reflexive, $z(t)$ is differentiable a.e. for $t \in [0, T_1]$. Now, let $z(t)$ is differentiable at $t = t_0$ and $h > 0$. Putting $\tau = t_0$ and $t = t_0 + h$ in (2.2)

$$\begin{aligned}&\|z(t_0 + h) - x\| - \|z(t_0) - x\| \\ &\leq \int_{t_0}^{t_0+h} \{[z(\eta) - x, G(\eta, z_\eta) - y]_+ + \theta(\eta, t_0)\} d\eta\end{aligned}$$

for $[x, y] \in A(t_0, z_{t_0})$, where $\theta(\eta, t_0) = L_1(\|x\|)|t_0 - \eta|(1 + \|y\| + M)$. Dividing this by h and letting $h \downarrow 0$, it follows

$$[z(t_0) - x, z'(t_0)]_+ \leq [z(t_0) - x, G(t_0, z_{t_0}) - y]_+.$$

Therefore

$$[z(t_0) - x, -z'(t_0) + G(t_0, z_{t_0}) - y]_+ \geq 0$$

for $[x, y] \in A(t_0, z_{t_0})$. By the maximality of $A(t_0, z_{t_0})$,

$$z'(t_0) + A(t_0, z_{t_0})z(t_0) \ni G(t_0, z_{t_0}). \quad \square$$

THEOREM 5. *Any strong solution of (FDE) on $[0, T_1]$ is an integral solution.*

Proof. Let $z(t)$ be a strong solution of (FDE) and let $[x, y] \in A(\bar{r}, z_{\bar{r}})$, where $\bar{r} \in [0, T_1]$. We note that there exists $x^* \in F(z(t) - x)$ such that

$$x^*(z'(t) - y) = < z(t) - x, z'(t) - y >_s \text{ a.e.} t \in [0, T_1].$$

Without loss of generality, we assume that $z(t)$ is differentiable at t. Then using [8,Lemma 1.3] for each $x^* \in F(z(t) - x)$

$$\begin{aligned}&\|z(t) - x\| \frac{d}{dt} \|z(t) - x\| = (z'(t), x^*) \\ &= (z'(t) + y - G(t, z_t), x^*) + (-y + G(t, z_t), x^*).\end{aligned}$$

Since there is $x^* \in F(z(t)-x)$ such that

$$(z'(t)+y-G(t,z_t),x^*) = < z(t)-x, z'(t)+y-G(t,z_t) >_i$$

and for all $x^* \in F(z(t)-x)$, $[x,y] \in A(\bar{r}, z_{\bar{r}})$,

$$(-y+G(t,z_t),x^*) \leq < z(t)-x, -y+G(t,z_t) >_s,$$

$$(2.4)\quad \frac{d}{dt}\|z(t)-x\| = [z(t)-x, z'(t)-G(t,z_t)+y]_- + [z(t)-x, -y+G(t,z_t)]_+.$$

On the other hand,

$$\begin{aligned}\|z(t)-x\| &= \|J_\lambda(t,z_t)(z(t)+\lambda(-z'(t)+G(t,z_t))) \\ &\quad - J_\lambda(\bar{r},z_{\bar{r}})(x+\lambda y)\| \\ &\leq \|J_\lambda(t,z_t)(z(t)+\lambda(-z'(t)+G(t,z_t))) \\ &\quad - J_\lambda(t,z_t)(x+\lambda y)\| \\ &\quad + \|J_\lambda(t,z_t)(x+\lambda y) - J_\lambda(\bar{r},z_{\bar{r}})(x+\lambda y)\|.\end{aligned}$$

By (A.2),

$$\Big[\|z(t)-x\| - \|(z(t)-x)+\lambda(-z'(t)+G(t,z_t)-y)\|\Big]/\lambda \leq L_1(\|x+\lambda y\|)|t-\bar{r}|(1+\|y\|+\bar{M}),$$

where $\bar{M}$ is the Lipschitz constant of the strong solution $z(t)$ of (FDE) on $[0,T_1]$. Letting $\lambda \downarrow 0$ with this,

$$[z'(t)-x, y-(-z'(t)+G(t,z_t))]_- \leq L_1(\|x\|)|t-\bar{r}|(1+\|y\|+\bar{M}).$$

Therefore we have

$$\frac{d}{dt}\|z(t)-x\| \leq L_1(\|x\|)(1+\|y\|+\bar{M})|t-\bar{r}| + [z(t)-x, G(t,z_t)-y]_+.$$

Integrating this in $[\tau,t]$,

$$\|z(t)-x\| - \|z(\tau)-x\| \leq \int_\tau^t \{[z(\eta)-x, G(\eta,z_\eta)-y]_+ + C|\eta-\bar{r}|\}d\eta,$$

for $\bar{r} \in [0,T_1]$, $[x,y] \in A(\bar{r}, z_{\bar{r}})$, and some nonnegative constant C. □

REMARK 1. Theorem 4 and Theorem 5 show that (FDE) has a unique strong solution in a reflexive Banach space X.

REMARK 2. It is obious from the proof of the above theorems that the interval $[0,T]$ can be replaced by $[T_1,T]$. Then the solution $z(t)$ of (FDE) exists beyond T_1. With this processing, we may conclude that there exists a maximal interval of existence of solutions of (FDE) on $[0,T_1]$.

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