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CONFORMAL CHANGE OF THE TENSOR $U^{\nu}{}_{\lambda\mu}$ IN 6–DIMENSIONAL g–UFT

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I. INTRODUCTION.

The conformal change in a generalized 4-dimensional Riemannian space connected by an Einstein's connection was primarily studied by HLAVATÝ([8], 1957). CHUNG ([6], 1968) also investigated the same topic in 4-dimensional *g-unified field theory.

The Einstein's connection induced by the conformal change for all classes in 3-dimensional case, for the second and third classes in 5-dimensional case, and for the first class in 5-dimensional case were investigated by CHO([1], 1992), ([2], 1994). CHO([3], 1995) also studied change of the torsion tension $S_{w\mu}{}^{\nu}$ induced by the conformal change for the second class with the first category in 6-dimensional g-unified field theory.

In the present paper, we investigate change of the tensor $U^{\nu}{}_{\lambda\mu}$ induced by the conformal change in 6-dimensional g-unified field theory. These topics will be studied for the second class with the first category in 6-dimensional case.

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II. PRELIMINARIES.

This chapter is a brief collection of basic concepts, notations, theorems, and results needed in our further considerations. They may be refferd to CHUNG([5], 1982; [4], 1988), CHO([1], 1992; [2], 1994; [3], 1995).

2.1. n-dimensional g-unified field theory.

The n-dimensional g-unified field theory (n-g-UFT hereafter) was originally suggested by HLAVATÝ([8], 1957) and systematically introduced by CHUNG([7], 1963).

Let X_n[1] be an n-dimensional generalized Riemannian manifold, referred to a real coordinate system x^ν obeying coordinate transformations $x^\nu \rightarrow x^{\nu'}$, for which

$$\text{Det}\left(\left(\frac{\partial x}{\partial x'}\right)\right) \neq 0. \tag{2.1}$$

In the usual Einstein's n-dimensional unified field theory, the manifold X_n is endowed with a general real nonsymmetric tensor $g_{\lambda\mu}$ which may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$[2] :

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu} \tag{2.2}$$

where

$$\text{Det}((g_{\lambda\mu})) \neq 0, \qquad \text{Det}((h_{\lambda\mu})) \neq 0. \tag{2.3}$$

[1] Throughout the present paper, we assumed that $n \geq 2$.

[2] Throughout this paper, Greek indices are used for holonomic components of tensors. In X_n all indices take the values $1, \cdots, n$ and follow the summation convention.

Therefore we may define a unique tensor $h^{\lambda\nu} = h^{\nu\lambda}$ by

$$h_{\lambda\mu}h^{\lambda\nu} = \delta^{\nu}_{\mu}. \tag{2.4}$$

In our n-g-UFT, the tensors $h_{\lambda\mu}$ and $h^{\lambda\nu}$ will serve for raising and/or lowering indices of the tensors in X_n in the usual manner.

The manifold X_n is connected by a general real connection $\Gamma^{\nu}_{\omega\mu}$ with the following transformation rule :

$$\Gamma^{\nu'}_{\omega'\mu'} = \frac{\partial x^{\nu'}}{\partial x^{\alpha}}\left(\frac{\partial x^{\beta}}{\partial x^{\omega'}} \cdot \frac{\partial x^{\gamma}}{\partial x^{\mu'}}\Gamma^{\alpha}_{\beta\gamma} + \frac{\partial^2 x^{\alpha}}{\partial x^{\omega'}\partial x^{\mu'}}\right) \tag{2.5}$$

and satisfies the system of Einstein's equations

$$D_w g_{\lambda\mu} = 2S_{w\mu}{}^{\alpha}g_{\lambda\alpha} \tag{2.6}$$

where D_w denotes the covariant derivative with respect to $\Gamma^{\nu}_{\lambda\mu}$ and

$$S_{\lambda\mu}{}^{\nu} = \Gamma^{\nu}_{[\lambda\mu]} \tag{2.7}$$

is the *torsion tensor* of $\Gamma^{\nu}_{\lambda\mu}$. The connection $\Gamma^{\nu}_{\lambda\mu}$ satisfying (2.6) is called the *Einstein's connection.*

In our further considerations, the following scalars, tensors, abbreviations, and notations for $p = 0, 1, 2, \cdots$ are frequently used :

(2.8)a
$$\mathfrak{g} = \mathrm{Det}((g_{\lambda\mu})) \neq 0,\ \mathfrak{h} = \mathrm{Det}((h_{\lambda\mu})) \neq 0,$$
$$\mathfrak{k} = \mathrm{Det}((k_{\lambda\mu})),$$

(2.8)b
$$g = \frac{\mathfrak{g}}{\mathfrak{h}}, \quad k = \frac{\mathfrak{k}}{\mathfrak{h}},$$

(2.8)c
$$K_p = k_{[\alpha_1}{}^{\alpha_1} \cdots k_{\alpha_p]}{}^{\alpha_p}, \quad (p = 0, 1, 2, \cdots)$$

(2.8)d
$${}^{(0)}k_{\lambda}{}^{\nu} = \delta^{\nu}_{\lambda},\ {}^{(1)}k_{\lambda}{}^{\nu} = k_{\lambda}{}^{\nu},\ {}^{(p)}k_{\lambda}{}^{\nu} = {}^{(p-1)}k_{\lambda}{}^{\alpha}k_{\alpha}{}^{\nu},$$

(2.8)e
$$K_{\omega\mu\nu} = \nabla_\nu k_{\omega\mu} + \nabla_\omega k_{\nu\mu} + \nabla_\mu k_{\omega\nu},$$

(2.8)f
$$\sigma = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}.$$

where ∇_ω is the symbolic vector of the convariant derivative with respect to the Christoffel symbols $\{{}^{\nu}_{\lambda\mu}\}$ defined by $h_{\lambda\mu}$. The scalars and vectors introduced in (2.8) satisfy

(2.9)a
$$K_0 = 1;\ K_n = k \text{ if } n \text{ is even};\ K_p = 0 \text{ if } p \text{ is odd},$$

(2.9)b
$$g = 1 + K_2 + \cdots + K_{n-\sigma},$$

(2.9)c
$${}^{(p)}k_{\lambda\mu} = (-1)^{p\,(p)}k_{\mu\lambda}, \quad {}^{(p)}k^{\lambda\nu} = (-1)^{p\,(p)}k^{\nu\lambda}.$$

Furthermore, we also use the following useful abbreviations, denoting an arbitrary tensor $T_{\omega\mu\nu}$, skew-symmetric in the first two indices, by T :

(2.10)a
$$\overset{pqr}{T} = \overset{pqr}{T}{}_{\omega\mu\nu} = T_{\alpha\beta\gamma}\,{}^{(p)}k_\omega{}^{\alpha\,(q)}k_\mu{}^{\beta\,(r)}k_\nu{}^\gamma,$$

(2.10)b
$$T = T_{\omega\mu\nu} = \overset{000}{T},$$

(2.10)c
$$2\overset{pqr}{T}{}_{\omega[\lambda\mu]} = \overset{pqr}{T}{}_{\omega\lambda\mu} - \overset{pqr}{T}{}_{\omega\mu\lambda},$$

(2.10)d
$$2\overset{(pq)r}{T}{}_{\omega\lambda\mu} = \overset{pqr}{T}{}_{\omega\lambda\mu} + \overset{qpr}{T}{}_{\omega\lambda\mu}.$$

We then have

(2.11)
$$\overset{pqr}{T}{}_{\omega\lambda\mu} = -\overset{qpr}{T}{}_{\lambda\omega\mu}.$$

If the system (2.6) admits $\Gamma^{\nu}_{\lambda\mu}$, using the above abbreviations it was shown that the connection is of the form

$$\Gamma^{\nu}_{\omega\mu} = \{{}^{\nu}_{\omega\mu}\} + S_{\omega\mu}{}^{\nu} + U^{\nu}{}_{\omega\mu} \tag{2.12}$$

where

$$U_{\nu\omega\mu} = \overset{100}{S}{}_{(\omega\mu)\nu} + \overset{(10)0}{S}{}_{\nu(\omega\mu)}. \tag{2.13}$$

The above two relations show that *our problem of determining $\Gamma^{\nu}_{\omega\mu}$ in terms of $g_{\lambda\mu}$ is reduced to that of studying the tensor $S_{\omega\mu}{}^{\nu}$*. On the other hand, it has also been shown that the tensor $S_{\omega\mu}{}^{\nu}$ satisfies

$$S = B - 3\overset{(110)}{S} \tag{2.14}$$

where

$$2B_{\omega\mu\nu} = K_{\omega\mu\nu} + 3K_{\alpha[\mu\beta}k_{\omega]}{}^{\alpha}k_{\nu}{}^{\beta}. \tag{2.15}$$

2.2. Some results in 6-g-UFT.

In this section, we introduce some results of 6-g-UFT without proof, which are needed in our subsequent considerations.

DEFINITION (2.1). In 6-g-UFT, the tensor $g_{\lambda\mu}(k_{\lambda\mu})$ is said to be of the second class with the first category, if $K_2 \neq 0$, $K_4 = K_6 = 0$.

THEOREM (2.2). *(Main recurrence relation) In X_6, the following recurrence relation holds :*

(Second class with the first category)

$$^{(p+2)}k_{\lambda}{}^{\nu} = -K_2\,{}^{(p)}k_{\lambda}{}^{\nu}, \quad (p = 1, 2, \cdots) \tag{2.16}$$

THEOREM (2.3). *(For the second class with the first category in 6-g-UFT). A necessary and sufficient condition for the existence and uniqueness of the solution of (2.5) is*

$$1-(K_2)^2 \neq 0. \tag{2.17}$$

If the condition (2.17) is satisfied, the unique solution of (2.14) is given by

$$(1-(K_2)^2)(B-S) = K_2(1-K_2)B + 2\overset{(10)1}{B}. \tag{2.18}$$

III. CONFORMAL CHANGE OF THE 6-DIMENSIONAL TENSOR $U^{\nu}{}_{\lambda\mu}$ FOR THE SECOND CLASS WITH THE FIRST CATEGORY.

In this final chapter we investigate the change $U^{\nu}{}_{\lambda\mu} \to \overline{U}^{\nu}{}_{\lambda\mu}$ of the tensor induced by the conformal change of the tensor $g_{\lambda\mu}$, using the recurrence relations and theorems introduced in the preceding chapter.

We say that X_n and $\overline{X}_n$ are conformal if and only if

$$\overline{g}_{\lambda\mu}(x) = e^{\Omega} g_{\lambda\mu}(x) \tag{3.1}$$

where $\Omega = \Omega(x)$ is an at least twice differentiable function. This conformal change enforces a change of the tensor $U^{\nu}{}_{\lambda\mu}$. An explicit representation of the change of 6-dimensional tensor $U^{\nu}{}_{\lambda\mu}$ for the second class with the first category will be exhibited in this chapter.

AGREEMENT (3.1). Throughout this section, we agree that, if T is a function of $g_{\lambda\mu}$, then we denote $\overline{T}$ the same function of $\overline{g}_{\lambda\mu}$. In particular, if T is a tensor, so is $\overline{T}$. Furthermore, the indices of T ($\overline{T}$) will be raised and/or lowered by means of $h^{\lambda\nu}$ ($\overline{h}^{\lambda\nu}$) and/or $h_{\lambda\mu}$ ($\overline{h}_{\lambda\mu}$).

The results in the following theorems are needed in our further considerations. They may be referred to CHO([1], 1992; [2], 1994; [3], 1995).

THEOREM (3.2). *In n-g-UFT, the conformal change (3.1) induces the following changes :*

$$(3.2)a \qquad \begin{aligned} {}^{(p)}\overline{k}_{\lambda\mu} &= e^{\Omega(p)}\,{}^{(p)}k_{\lambda\mu}, \quad {}^{(p)}\overline{k}_{\lambda}{}^{\nu} = {}^{(p)}k_{\lambda}{}^{\nu}, \\ {}^{(p)}\overline{k}^{\lambda\nu} &= e^{-\Omega(p)}\,{}^{(p)}k^{\lambda\nu} \end{aligned}$$

$$(3.2)b \qquad \overline{g} = g, \quad \overline{K_p} = K_p, \qquad (p = 1, 2, \cdots).$$

Now, we are ready to derive representations of the changes $U^{\nu}{}_{\lambda\mu} \to \overline{U}^{\nu}{}_{\lambda\mu}$ in 6-g-UFT for the second class with the first category induced by the conformal change (3.1).

THEOREM (3.3). *(For the second class with the first category) The change $S_{w\mu}{}^{\nu} \to \overline{S}_{w\mu}{}^{\nu}$ induced by conformal change (3.1) may be represented by*

$$(3.3) \qquad \begin{aligned} \overline{S}_{w\mu}{}^{\nu} =& S_{w\mu}{}^{\nu} + \frac{1}{1-(K_2)^2}\Big(-4K_2\,{}^{(2)}k^{\nu}{}_{[w}k_{\mu]}{}^{\delta}\Omega_{\delta} \\ & + (1-2K_2)k^{\nu}{}_{[w}\Omega_{\mu]} + (1+2K_2)k^{\nu}{}_{[w}{}^{(2)}k_{\mu]}{}^{\delta}\Omega_{\delta}\Big) \\ & + \frac{1}{1+K_2}\big(-k_{w\mu}\Omega^{\nu} - h^{\nu}{}_{[w}k_{\mu]}{}^{\delta}\Omega_{\delta} + k_{w\mu}{}^{(2)}k^{\nu\delta}\Omega_{\delta}\big), \end{aligned}$$

where $\Omega_{\mu} = \partial_{\mu}\Omega$.

THEOREM (3.4). *The change $U^{\nu}{}_{\lambda\mu} \to \overline{U}^{\nu}{}_{\lambda\mu}$ induced by the conformal change (3.1) may be represented by*

$$(3.4) \qquad \begin{aligned} \overline{U}^{\nu}{}_{\lambda\mu} =& U^{\nu}{}_{\lambda\mu} + \frac{1}{1-(K_2)^2}\Big\{(2-9K_2)\,{}^{(2)}k^{\nu}{}_{(\lambda}{}^{(2)}k_{\mu)}{}^{\delta} \\ & + (3+K_2)\,{}^{(2)}k_{\lambda\mu}{}^{(2)}k^{\nu\delta} + 2(2(K_2)^2-1)k^{\nu}{}_{(\lambda}k_{\mu)}{}^{\delta}\Big\}\Omega_{\delta} \\ & + \frac{1}{1-K_2}\Big\{2\,{}^{(2)}k^{\nu}{}_{(\lambda}\Omega_{\mu)} - \frac{1+4K_2}{1+K_2}\,{}^{(2)}k_{\lambda\mu}\Omega^{\nu} \\ & - 2k^{\nu}{}_{(\lambda}{}^{(2)}k_{\mu)}{}^{\delta}\Omega_{\delta}\Big\} + \frac{1}{1+K_2}h_{\lambda\mu}{}^{(2)}k^{\nu\delta}\Omega_{\delta}. \end{aligned}$$

Proof. In virtue of (2.13) and Agreement (3.1), we have

$$\overline{U}_{\nu\lambda\mu} = \overset{\overline{100}}{S}_{(\lambda\mu)\nu} + \overset{\overline{(10)0}}{S}_{\nu(\lambda\mu)}. \tag{3.5}$$

The relation (3.4) follows by substituiting (3.3), (2.10), Definition (2.1), (2.16), (3.2) into (3.5). □

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