Kangweon–Kyungki Math. Jour. **3** (1995), No. 1, pp. 81–88

ON 4–DIMENSIONAL EINSTEIN MANIFOLDS WHICH ARE POSITIVELY PINCHED

KWAN–SEOK KO

1. Introduction.

The purpose of this paper is to prove the following.

THEOREM. *Let M be a closed oriented connected Einstein 4-manifold whose sectional curvature k satisfies $1 \geq k \geq \delta$. If $\delta \geq \frac{2}{17} \approx$ 0.1176471, then M is topologically S^4 or $\pm\mathbb{C}P^2$.*

In [10], Seaman proved that if $\delta \geq \frac{1}{3\vdots\ 1+3\cdot 2^{\frac{1}{4}}/5^{\frac{1}{2}}+1} \approx 0.1714$ then δ pinched Riemannin 4 manifold is definite. Under this Seaman's pinching condition, we obtained

$$|\sigma(M)| < \frac{1}{2}\chi(M), \tag{1}$$

where $\sigma(M)$ is the signature of M and $\chi(M)$ is the Euler characteristic of M [7]. It follows that the second Betti number satisfies $b_2(M) \leq 1$. Since M is compact, even dimensional, oriented, and positively curved, it is simply connected by Synge theorem. Freedman's classification theorem [5] states that smooth compact simply connected 4-manifolds are classified topologically by their intersection form. Therefore M is topologically a 4-sphere S^4 or a complex projective 2-plane $\pm\mathbb{C}P^2$.

We will apply Seaman's method to the Einstein manifold with pinching hypothesis. This idea was originally due to Berger [3] in dimension 5 (later, Bourguignon [4] in dimension 4). First, we show that the manifold is definite under the hypothesis of theorem, and then by adapting Hitchin's argument [6, 11] we have the same inequality (1). Then we have the conclusion of theorem.

Received October 17, 1994.

We announce here that the pinching constant in the theorem lower down to 0.1138113.

For a compact Einstein 4-manifold with nonnegative (or nonpositive) sectional curvature, Hitchin [6] proved that

$$|\sigma(M)| \leq \left(\frac{2}{3}\right)^{\frac{3}{2}} \chi(M) \approx 0.5543\chi(M).$$

2. Harmonic 2 form and Weitzenböck formula.

A harmonic 2-form X on a Riemannian manifold satisfies

$$0 = \frac{1}{2}\Delta|X|^2 + \nabla|X|^2 + < R_2X, X >, \tag{2}$$

where R_2 is the Weitzenböck operator whose pointwise action is as follows:

$$\begin{aligned} &< R_2(v_1 \wedge v_2), w_1 \wedge w_2) > \\ =&\mathrm{Ric}(v_1, w_1) < v_2, w_2 > + \mathrm{Ric}(v_2, w_2) < v_1, w_1 > \\ &- \mathrm{Ric}(v_1, w_2) < v_2, w_1 > - \mathrm{Ric}(v_2, w_1) < v_1, w_2 > \\ &- 2 < R(v_1, v_2)w_1, w_2 >, \end{aligned} \tag{3}$$

where v_i, w_i are tangent vectors, $< \cdot, \cdot >$ is the inner product, Ric(resp. R) is the Ricci (resp. Riemann) curvature tensor and we identify two vectors with two forms via the inner product.

On an oriented 4-manifold M, one has the Hodge star operator $*$ taking two forms to two forms and satisfying $*^2 = 1$, which yields the splitting of these forms into the ± 1 eigenspace $\Lambda^2_{\pm}$.

Given $X_{\pm} \in \Lambda^2_{\pm}$, at any point p, there are orthonormal vectors e_1, e_2 such that

$$\frac{X_+}{|X_+|} + \frac{X_-}{|X_-|} = \sqrt{2}e_1 \wedge e_2.$$

Let T_pM be the tangent space of M at a fixed point $p \in M$.

Letting $\{e_1, e_2, e_3, e_4\}$ be a positively oriented orthonormal basis for T_pM, we have $*(e_1 \wedge e_2) = e_3 \wedge e_4$. Let X be a two-form on a four-manifold with X_+, X_- the self-dual and anti-self-dual components, respectively. Then for $X = X_+ + X_-$, we get

$$X = \frac{\sqrt{2}}{2}(|X_+| + |X_-|)e_1 \wedge e_2 + \frac{\sqrt{2}}{2}(|X_+| - |X_-|)e_3 \wedge e_4 \quad \text{at} \quad p.$$

Using (3), it is easy to see that

$$< R_2X, X >_p = |X|^2(K_{13}+K_{14}+K_{23}+K_{24}) - 2R_{1234}(|X_+|^2 - |X_-|^2),$$

where $K_{ij} = (K(P))$ is the sectional curvature of the plane $e_i \wedge e_j (= P)$ and

$$R_{ijkl} =< R(e_i, e_j)e_k, e_l > .$$

If we assume that $1 \geq k \geq \delta$, then using the Berger's inequality $|R_{ijkl}| \leq \frac{2}{3}(1-\delta)$, we have a global estimate

$$(4) \qquad < R_2X, X > \geq 4\delta|X|^2 - \frac{4}{3}(1-\delta)\left||X_+|^2 - |X_-|^2\right|.$$

Kato inequality states that if $X_p \neq 0$, then one has $|\nabla|X||^2 \leq |\nabla X|^2$ at p.

On applying the conformal invariance of the middle dimensional harmonic forms, Seamam [10] improved Kato's inequality for 4-manifolds in the following.

PROPOSITION 1([10]). *Let M be a 4-dimensional Riemannian manifold. Let X be a harmonic 2-form on M. Then X satisfies the pointwise inequality*

$$(5) \qquad |\nabla X|^2 \geq \frac{3}{2}|\nabla|X||^2.$$

In order to obtain the theorem, we need the estimate about the first eigenvalue λ_1 of the Laplacian acting on functions of M.

PROPOSITION 2(LICHNEROWICZ[1]). *If the Ricci tensor R_{ij} of M, a compact Riemannian n-dimensional manifold with the metric tensor g_{ij}, is such that 2-tensor $R_{ij} - kg_{ij}$ is nonnegative for some $k > 0$, then $\lambda_1 \geq \frac{nk}{n-1}$.*

PROPOSITION 3. *Under the hypothesis of theorem, the first nonzero eigenvalue λ_1 of the Laplacian action on functions of M satisfies*

$$(6) \qquad \lambda_1 \geq \frac{4+8\delta}{3}.$$

Proof. A 4-dimensional Einstein metric is a metric for which the Ricci tensor R_{ij} and the metric tensor g_{ij} is proportional

$$R_{ij} = \frac{S}{4} g_{ij}$$

where the scalar curvature S is a constant.

Singer and Thorpe's characterization of an Einstein manifold is that, for each tangent plane P, $K(P) = K(P^{\perp})$ where $P^{\perp}$ is the oriented orthogonal complement of P [11]. Hence $S \geq 4 + 8\delta$. Combining the Proposition 2, we get

$$\lambda_1 \geq \frac{4}{3} \cdot \frac{S}{4} \geq \frac{4 + 8\delta}{3}. \qquad \square$$

3. Euler characteristic and Signature.

We use the normal form for the curvature tensor at each point of a 4-dimensional Einstein manifold.

We regard the curvature tensor R as a self-adjoint linear endomorphism of the bundle Λ^2 of 2-forms defined by

$$R(e_i \wedge e_j) = \frac{1}{2} \sum R_{ijkl} e_k \wedge e_l$$

relative to a local orthonormal basis $\{e_i\}$ of the 1 forms.

The Einstein curvature tensor R can be decomposed into the orthogonal components which have the same symmetries as R :

$$R = U + W$$

where W is the Weyl conformal curvature tensor, and U denote the scalar curvature part. Singer and Thorpe showed that $*W* = W$. Therefore W decompose into $W^{\pm}$. Here W^+ and W^- are the self-dual and anti-self-dual components of W, respectively.

The theorem on the normal form of R states that there exists an orthonormal basis such that relative to the corresponding basis $\{e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_3 \wedge e_4, e_4 \wedge e_2, e_2 \wedge e_3\}$ of Λ^2, R takes the form

$$\begin{bmatrix} A & B \\ B & A \end{bmatrix}$$

where $A = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$, $B = \begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{bmatrix}$. The Bianchi identity implies that

$\sum_{i=1}^{3} b_i = 0$, moreover $\sum_{i=1}^{3} a_i = \frac{1}{2}\mathrm{Trace}(R) = \frac{S}{4}$. It will be convenient to regard $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ as vectors $\vec{a}, \vec{b} \in \mathbb{R}^3$.

We make use of this expression of R to write the formulas for the integrands of the characteristic numbers :

$$\begin{aligned}
\chi(M) &= \int_M \chi(R)dV = \frac{1}{8\pi^2}\int_M \|U\|^2 + \|W\|^2 dV, \\
\sigma(M) &= \frac{1}{12\pi^2}\int_M \|W^+\|^2 - \|W^-\|^2 dV, \\
\sigma(M) &= \frac{1}{3}p_1(M) = \frac{1}{3}\int_M p_1(R)dV,
\end{aligned}$$

where $p_1(M)$ is the Pontryagin number of M.

With these notations above, we have

$$\begin{aligned}
\|U + W\|^2 &= 2\sum_{i=1}^{3}(a_i^2 + b_i^2), \\
\|W^+\|^2 &= \sum_{i=1}^{3}(a_i' + b_i)^2, \text{ where } a_i' = a_i - \alpha,\ U = \alpha Id_{S^2\Lambda^2 V}, \\
\|W^-\|^2 &= \sum_{i=1}^{3}(a_i' - b_i)^2, \\
\|W^+\|^2 - \|W^-\|^2 &= 4\sum_{i=1}^{3} a_i' b_i = 4\sum_{i=1}^{3} a_i b_i \quad (\text{because } \sum_{i=1}^{3} b_i = 0).
\end{aligned}$$

We obtain

$$\chi(R) - p_1(R) = \frac{1}{4\pi^2}\left(2\sum_{i=1}^{3}(a_i^2 + b_i^2) - 4\sum_{i=1}^{3} a_i b_i\right). \tag{7}$$

4. Proof of theorem.

We first show that the manifold satisfying the hypothesis of theorem is definite.

Assume that M is indefinite, then there exist nonzero harmonic 2-forms X_+ and X_- so that

$$\int |X_+| = \int |X_-|.$$

Now $|X_\pm|$ are weakly differentiable [2]. From the variational characterization of λ_1, one has

$$\int |\nabla(|X_+| - |X_-|)|^2 \geq \lambda_1 \int (|X_+| - |X_-|)^2. \tag{8}$$

From (5), we get the following inequality in the sense of distribution

$$|\nabla(|X_+| - |X_-|)|^2 \leq \frac{4}{3}|\nabla X|^2.$$

Using this inequality, we obtain

$$\int |\nabla X|^2 \geq \frac{3}{4}\lambda_1 \int (|X_+| - |X_-|)^2. \tag{9}$$

We substitute (4), (6) and (9) into (2) and integrate over M. Since $\int_M \Delta|X|^2 = 0$, we have

$$0 \geq \int 4\delta|X|^2 - \frac{4}{3}(1-\delta)\,\big||X_+|^2 - |X_-|^2\big| + (1+2\delta)(|X_+| - |X_-|)^2. \tag{10}$$

Let $b = |X_+| + |X_-| \geq ||X_+| - |X_-|| = a$. We may write the integrand in (10) as

$$2\delta b^2 - \frac{4}{3}(1-\delta)ab + (1 + 2\delta + 2\delta)a^2.$$

If $a = 0$, then this is an obvious contradiction.

Suppose now that $a \neq 0$. A contradiction to M's indefiniteness is obtained if we can show that

$$6\delta\alpha^2 - 4(1-\delta)\alpha + 3(1+4\delta) \geq 0 \quad \text{where } \alpha = \frac{b}{a} \geq 1. \tag{11}$$

It follows from the discriminant of (11), we have $(17\delta - 2)(2\delta + 1) \geq 0$. Thus we conclude that M is a definite manifold if $\delta \geq \frac{2}{17}$.

Next, we show that (1) holds for the Einstein manifold with given conditions. Since the critical values of sectional curvature are positive, the numbers $\{a_i\}_{i=1}^3$ are all positive. Observing (7), it is easy to majorize its quantity $\dfrac{2\sum_{i=1}^{3} a_i b_i}{\sum_{i=1}^{3}(a_i^2 + b_i^2)}$ when variables $\{a_i\}$, $\{b_i\}$ are subject to the constraints $\sum_{i=1}^{3} b_i = 0$ and $\delta \leq a_i \leq 1$, $i = 1, 2, 3$.

In the Euclidean space $\mathbb{R}^3$ consider two vectors $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$. Let θ be the angle of these two vectors $\vec{a}$ and $\vec{b}$. Then we set $\dfrac{2\sum_{i=1}^{3} a_i b_i}{\sum_{i=1}^{3}(a_i^2 + b_i^2)} \leq \cos\theta$. Therefore we have $\cos\theta \leq \sqrt{\frac{2}{3}} \frac{1-\delta}{\sqrt{1+2\delta^2}}$. In fact, $\sin\theta = \dfrac{a_1 + a_2 + a_3}{\sqrt{3}\sqrt{a_1^2 + a_2^2 + a_3^2}}$ attains its minimum on the boundary of the domain $\{\delta \leq a_i \leq 1 \mid i = 1, 2, 3\}$, whence $\sin\theta \geq \frac{1}{\sqrt{3}} \frac{1+2\delta}{\sqrt{1+2\delta^2}}$. Thus we obtain,

$$p_1(R) \leq 2\sqrt{\frac{2}{3}} \cdot \frac{1-\delta}{\sqrt{1+\delta^2}}.$$

This is equivalent to the inequality

$$|\sigma(M)| \leq \left(\frac{2}{3}\right)^{\frac{3}{2}} \frac{1-\delta}{\sqrt{1+2\delta^2}} \chi(M). \tag{12}$$

We conclude that (1) holds if $\delta \geq \frac{2}{17}$. □

REMARK 1. *We note that (1) holds for a negative $\frac{2}{17}$-pinched Einstein manifold.*

REMARK 2. *From (12) if we solve* $\left(\frac{2}{3}\right)^{\frac{3}{2}} \dfrac{1-\delta}{\sqrt{1+2\delta^2}} < \frac{1}{2}$, *then (1)* *is satisfied for the Einstein 4-manifold with* $1 \geq k \geq \delta \approx 0.0761326$ *(or* $-1 \leq k \leq -\delta \approx -0.0761326$*).*

References

[1] T. Aubin,, *Nonlinear analysis on manifolds, Monge-Ampére equation*, Springer Verlag, New York, 1982.

[2] P. Bérard, *"Spectral geometry : Direct and inverse problem"*, Lecture Note in Math. **1207** (1986), Springer Verlag, New York.

[3] M. Berger, *Sur les variétés 4/23-pincées de dimension 5*, C.R. Acad. Sci. Paris **257** (1963), 4122-4125.

[4] J.P. Bourguignon, *La conjecture de Hopf sur* $S^2 \times S^2$, *Géometrie reimannienne en dimension 4*, Seminar Arthur Besse, CEDIC Paris (1981), 747–355.

[5] M. Freedman, *The topology of four-dimensional manifolds*, J. Diff. Geom. **17** (1982), 327–454.

[6] N.J. Hitchin, *Compact four-dimensional Einstein manifolds*, Jour. Diff. Geom. **9** (1974), 463–466.

[7] Kwan-Seok Ko, *A pinching theorem for Riemannian 4-manifold*, Preprint.

[8] W. Seaman, *On four manifolds which are positively pinched*, Ann. Global Anal. Geom. **5. No.3** (1987), 193–198.

[9] W. Seaman, *A pinching theorem for four manifolds*, Geom. Dedicata **3** (1989), 37–40.

[10] ———, *Harmonic two forms in four-dimensions*, Proc. Amer. Math. Soc. **112. No.2** (1991), 545–548.

[11] P. Sentanac, *Le tenseur de courbure en dimension 4*, Seminar Arthur Besse, CEDIC Paris (1981), 203–219.

[12] I.M. Singer and J. A. Thorpe, *The curvature of 4-dimensional Einstein space*, Global analysis, papers in honor of K. Kodaira (1969), 355–365, Princeton University Press, Princeton.

Department of Mathematics
Inha University
Incheon, 402–751, Korea