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CONFORMAL CHANGE OF THE TORSION TENSOR IN 6–DIMENSIONAL g–UNIFIED FIELD THEORY

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I. INTRODUCTION

The conformal change in a generalized 4-dimensional Riemannian space connected by an Einstein's connection was primarily studied by HLAVATÝ([8], 1957). CHUNG ([6], 1968) also investigated the same topic in 4-dimensional $^*g$-unified field theory.

The Einstein's connection induced by the conformal change for all classes in 3-dimensional case, for the second and third classes in 5-dimensional case, and for the first class in 5-dimensional case were investigated by CHO([1], 1992), ([2], 1994).

In the present paper, we investigate change of the torsion tensor $S_{w\mu}{}^{\nu}$ induced by the conformal change in 6-dimensional g-unified field theory. These topics will be studied for the second class with the first category in 6-dimensional case.

II. PRELIMINARIES

This chapter is a brief collection of basic concepts, notations, theorems, and results needed in our further considerations. They may be refferrd to CHUNG([4], 1982; [3], 1988), CHO([1], 1992, [2], 1994).

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2.1. n-dimensional g-unified field theory.

The n-dimensional g-unified field theory (n-g-UFT hereafter) was originally suggested by HLAVATÝ([8], 1957) and systematically introduced by CHUNG([7], 1963).

Let X_n[1] be an n-dimensional generalized Riemannian manifold, referred to a real coordinate system x^ν obeying coordinate transformations $x^\nu \to x^{\nu'}$, for which

$$\text{Det}\left(\left(\frac{\partial x}{\partial x'}\right)\right) \neq 0. \tag{2.1}$$

In the usual Einstein's n-dimensional unified field theory, the manifold X_n is endowed with a general real nonsymmetric tensor $g_{\lambda\mu}$ which may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$[2] :

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu} \tag{2.2}$$

where

$$\text{Det}((g_{\lambda\mu})) \neq 0, \qquad \text{Det}((h_{\lambda\mu})) \neq 0. \tag{2.3}$$

Therefore we may define a unique tensor $h^{\lambda\nu} = h^{\nu\lambda}$ by

$$h_{\lambda\mu}h^{\lambda\nu} = \delta^\nu_\mu. \tag{2.4}$$

In our n-g-UFT, the tensors $h_{\lambda\mu}$ and $h^{\lambda\nu}$ will serve for raising and/or lowering indices of the tensors in X_n in the usual manner.

The manifold X_n is connected by a general real connection $\Gamma^\nu_{\omega\mu}$ with the following transformation rule :

$$\Gamma^{\nu'}_{\omega'\mu'} = \frac{\partial x^{\nu'}}{\partial x^\alpha}\left(\frac{\partial x^\beta}{\partial x^{\omega'}} \cdot \frac{\partial x^\gamma}{\partial x^{\mu'}}\Gamma^\alpha_{\beta\gamma} + \frac{\partial^2 x^\alpha}{\partial x^{\omega'}\partial x^{\mu'}}\right) \tag{2.5}$$

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[1] Throughout the present paper, we assumed that $n \geq 2$.

[2] Throughout this paper, Greek indices are used for holonomic components of tensors. In X_n all indices take the values $1, \cdots, n$ and follow the summation convention.

and satisfies the system of Einstein's equations

$$D_w g_{\lambda\mu} = 2S_{w\mu}{}^{\alpha} g_{\lambda\alpha} \tag{2.6}$$

where D_w denotes the covariant derivative with respect to $\Gamma^{\nu}_{\lambda\mu}$ and

$$S_{\lambda\mu}{}^{\nu} = \Gamma^{\nu}_{[\lambda\mu]} \tag{2.7}$$

is the *torsion tensor* of $\Gamma^{\nu}_{\lambda\mu}$. The connection $\Gamma^{\nu}_{\lambda\mu}$ satisfying (2.6) is called the *Einstein's connection.*

In our further considerations, the following scalars, tensors, abbreviations, and notations for $p = 0, 1, 2, \cdots$ are frequently used :

(2.8)a
$$\mathfrak{g} = \mathrm{Det}((g_{\lambda\mu})) \neq 0, \ \ \mathfrak{h} = \mathrm{Det}((h_{\lambda\mu})) \neq 0,$$
$$\mathfrak{k} = \mathrm{Det}((k_{\lambda\mu})),$$

(2.8)b
$$g = \frac{\mathfrak{g}}{\mathfrak{h}}, \quad k = \frac{\mathfrak{k}}{\mathfrak{h}},$$

(2.8)c
$$K_p = k_{[\alpha_1}{}^{\alpha_1} \cdots k_{\alpha_p]}{}^{\alpha_p}, \quad (p = 0, 1, 2, \cdots)$$

(2.8)d
$${}^{(0)}k_{\lambda}{}^{\nu} = \delta^{\nu}_{\lambda}, \ {}^{(1)}k_{\lambda}{}^{\nu} = k_{\lambda}{}^{\nu}, \ {}^{(p)}k_{\lambda}{}^{\nu} = {}^{(p-1)}k_{\lambda}{}^{\alpha} k_{\alpha}{}^{\nu},$$

(2.8)e
$$K_{\omega\mu\nu} = \nabla_{\nu} k_{\omega\mu} + \nabla_{\omega} k_{\nu\mu} + \nabla_{\mu} k_{\omega\nu},$$

(2.8)f
$$\sigma = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}.$$

where ∇_{ω} is the symbolic vector of the convariant derivative with respect to the Christoffel symbols $\{{}^{\nu}_{\lambda\mu}\}$ defined by $h_{\lambda\mu}$. The scalars and vectors introduced in (2.8) satisfy

(2.9)a
$$K_0 = 1; \ K_n = k \text{ if } n \text{ is even}; \ K_p = 0 \text{ if } p \text{ is odd},$$

(2.9)b $$g = 1 + K_2 + \cdots + K_{n-\sigma},$$

(2.9)c $${}^{(p)}k_{\lambda\mu} = (-1)^{p(p)}k_{\mu\lambda}, \quad {}^{(p)}k^{\lambda\nu} = (-1)^{p(p)}k^{\nu\lambda}.$$

Furthermore, we also use the following useful abbreviations, denoting an arbitrary tensor $T_{\omega\mu\nu}$, skew-symmetric in the first two indices, by T :

(2.10)a $$\overset{pqr}{T} = \overset{pqr}{T}_{\omega\mu\nu} = T_{\alpha\beta\gamma}{}^{(p)}k_{\omega}{}^{\alpha(q)}k_{\mu}{}^{\beta(r)}k_{\nu}{}^{\gamma},$$

(2.10)b $$T = T_{\omega\mu\nu} = \overset{000}{T},$$

(2.10)c $$2\overset{pqr}{T}_{\omega[\lambda\mu]} = \overset{pqr}{T}_{\omega\lambda\mu} - \overset{pqr}{T}_{\omega\mu\lambda},$$

(2.10)d $$2\overset{(pq)r}{T}_{\omega\lambda\mu} = \overset{pqr}{T}_{\omega\lambda\mu} + \overset{qpr}{T}_{\omega\lambda\mu}.$$

We then have

(2.11) $$\overset{pqr}{T}_{\omega\lambda\mu} = -\overset{qpr}{T}_{\lambda\omega\mu}.$$

If the system (2.6) admits $\Gamma^{\nu}_{\lambda\mu}$, using the above abbreviations it was shown that the connection is of the form

(2.12) $$\Gamma^{\nu}_{\omega\mu} = \{^{\ \nu}_{\omega\mu}\} + S_{\omega\mu}{}^{\nu} + U^{\nu}{}_{\omega\mu}$$

where

(2.13) $$U_{\nu\omega\mu} = \overset{100}{S}_{(\omega\mu)\nu} + \overset{(10)0}{S}_{\nu(\omega\mu)}.$$

The above two relations show that *our problem of determining* $\Gamma^{\nu}_{\omega\mu}$ *in terms of* $g_{\lambda\mu}$ *is reduced to that of studying the tensor* $S_{\omega\mu}{}^{\nu}$. On the other hand, it has also been shown that the tensor $S_{\omega\mu}{}^{\nu}$ satisfies

(2.14) $$S = B - 3\overset{(110)}{S}$$

where

(2.15) $$2B_{\omega\mu\nu} = K_{\omega\mu\nu} + 3K_{\alpha[\mu\beta}k_{\omega]}{}^{\alpha}k_{\nu}{}^{\beta}.$$

2.2. Some results in 6-g-UFT.

In this section, we introduce some results of 6-g-UFT without proof, which are needed in our subsequent considerations.

DEFINITION (2.1). In 6-g-UFT, the tensor $g_{\lambda\mu}(k_{\lambda\mu})$ is said to be :

(1) of the first class if $K_6 \neq 0$
(2) of the second class with the first category, if $K_2 \neq 0$, $K_4 = K_6 = 0$
(3) the second class with the second category, if $K_4 \neq 0$, $K_6 = 0$
(4) of the third class if

$$K_2 = K_4 = K_6 = 0.$$

Therefore, in 6-g-UFT we have four cases.

THEOREM (2.2). *(Main recurrence relations) In X_6, the following recurrence relations hold*

(First class)

(2.16)a
$$^{(p+6)}k_\lambda{}^\nu = -K_2\,{}^{(p+4)}k_\lambda{}^\nu - K_4\,{}^{(p+2)}k_\lambda{}^\nu - K_6\,{}^{(p)}k_\lambda{}^\nu, \quad (p = 0, 1, 2, \cdots)$$

(Second class with the second category)

$$(2.16)b \qquad ^{(p+4)}k_\lambda{}^\nu = -K_2\,{}^{(p+2)}k_\lambda{}^\nu - K_4\,{}^{(p)}k_\lambda{}^\nu, \quad (p = 0, 1, 2, \cdots)$$

(Second class with the first category)

$$(2.16)c \qquad ^{(p+2)}k_\lambda{}^\nu = -K_2\,{}^{(p)}k_\lambda{}^\nu, \quad (p = 1, 2, \cdots)$$

(Third class)

$$(2.16)d \qquad ^{(p)}k_\lambda^\nu = 0, \quad (p = 3, 4, \cdots).$$

THEOREM (2.3). *(For the second class with the first category in 6-g-UFT). A necessary and sufficient condition for the existence and uniqueness of the solution of (2.5) is*

$$1-(K_2)^2 \neq 0. \tag{2.17}$$

If the condition (2.17) is satisfied, the unique solution of (2.14) is given by

$$(1-(K_2)^2)(B-S) = K_2(1-K_2)B + 2\overset{(10)1}{B}. \tag{2.18}$$

III. CONFORMAL CHANGE OF THE 6-DIMENSIONAL TORSION TENSOR FOR THE SECOND CLASS WITH THE FIRST CATEGORY.

In this final chapter we investigate the change $S_{\lambda\mu}{}^{\nu} \rightarrow \overline{S}_{\lambda\mu}{}^{\nu}$ of the torsion tensor induced by the conformal change of the tensor $g_{\lambda\mu}$, using the recurrence relations and theorems introduced in the preceding chapter.

We say that X_n and $\overline{X}_n$ are conformal if and only if

$$\overline{g}_{\lambda\mu}(x) = e^{\Omega} g_{\lambda\mu}(x) \tag{3.1}$$

where $\Omega = \Omega(x)$ is an at least twice differentiable function. This conformal change enforces a change of the torsion tensor $S_{\lambda\mu}{}^{\nu}$. An explicit representation of the change of 6-dimensional torsion tensor $S_{\lambda\mu}{}^{\nu}$ for the second class with the first category will be exhibited in this chapter.

AGREEMENT (3.1). Throughout this section, we agree that, if T is a function of $g_{\lambda\mu}$, then we denote $\overline{T}$ the same function of $\overline{g}_{\lambda\mu}$. In particular, if T is a tensor, so is $\overline{T}$. Furthermore, the indices of T ($\overline{T}$) will be raised and/or lowered by means of $h^{\lambda\nu}$ ($\overline{h}^{\lambda\nu}$) and/or $h_{\lambda\mu}$ ($\overline{h}_{\lambda\mu}$).

The results in the following theorems are needed in our further considerations. They may be referred to CHO([1], 1992, [2], 1994).

THEOREM (3.2). *In n-g-UFT, the conformal change (3.1) induces the following changes :*

$$(3.2)a \qquad {}^{(p)}\overline{k}_{\lambda\mu} = e^{\Omega(p)}k_{\lambda\mu}, \quad {}^{(p)}\overline{k}_{\lambda}{}^{\nu} = {}^{(p)}k_{\lambda}{}^{\nu}, \quad {}^{(p)}\overline{k}^{\lambda\nu} = e^{-\Omega(p)}k^{\lambda\nu}$$

$$(3.2)b \qquad \overline{g} = g, \quad \overline{K_p} = K_p, \qquad (p = 1, 2, \cdots).$$

THEOREM (3.3). *(For all classes in 6-g-UFT). The change of the tensor $B_{\omega\mu\nu}$ induced by the conformal change (3.1) may be given by*

$$(3.3) \qquad \begin{aligned}\overline{B}_{\omega\mu\nu} =& e^{\Omega}(B_{\omega\mu\nu} + k_{\nu[\omega}\Omega_{\mu]} - k_{\omega\mu}\Omega_{\nu} \\ &- h_{\nu[\omega}k_{\mu]}{}^{\delta}\Omega_{\delta} + 2{}^{(2)}k_{\nu[\omega}k_{\mu]}{}^{\delta}\Omega_{\delta} + k_{\omega\mu}{}^{(2)}k_{\nu}{}^{\delta}\Omega_{\delta}).\end{aligned}$$

Now, we are ready to derive representations of the changes $S_{\omega\mu}{}^{\nu} \to \overline{S}_{\omega\mu}{}^{\nu}$ in 6-g-UFT for the second class with the first category induced by the conformal change (3.1).

THEOREM (3.4). *The conformal change (3.1) induces the following changes :*

$$(3.4) \qquad \begin{aligned} 2\,\overline{\overset{(10)1}{B}}{}_{\omega\mu\nu} =& e^{\Omega}[2\overset{(10)1}{B}{}_{\omega\mu\nu} + (-2{}^{(4)}k_{\nu[\omega}k_{\mu]}{}^{\delta} \\ &+ 2{}^{(2)}k_{\nu[\omega}k_{\mu]}{}^{\delta} - k_{\nu[\omega}{}^{(2)}k_{\mu]}{}^{\delta})\Omega_{\delta} - {}^{(3)}k_{\nu[\omega}\Omega_{\mu]}],\end{aligned}$$

THEOREM (3.5). *The change $S_{w\mu}{}^{\nu} \to \overline{S}_{w\mu}{}^{\nu}$ induced by conformal change (3.1) may be represented by*

$$(3.5) \qquad \begin{aligned}\overline{S}_{w\mu}{}^{\nu} =& S_{w\mu}{}^{\nu} + \frac{1}{1-(K_2)^2}(-4K_2^{(2)}k^{\nu}{}_{[w}k_{\mu]}{}^{\delta}\Omega_{\delta} \\ &+ (1-2K_2)k^{\nu}{}_{[w}\Omega_{\mu]} + (1+2K_2)k^{\nu}{}_{[w}{}^{(2)}k_{\mu]}{}^{\delta}\Omega_{\delta}) \\ &+ \frac{1}{1+K_2}(-k_{w\mu}\Omega^{\nu} - h^{\nu}{}_{[w}k_{\mu]}{}^{\delta}\Omega_{\delta} + k_{w\mu}{}^{(2)}k^{\nu\delta}\Omega_{\delta}),\end{aligned}$$

where $\Omega_\mu = \partial_\mu \Omega$.

Proof. In virtue of (2.18) and Agreement (3.1), we have

$$(1 - \overline{(K_2)}^2)(\overline{B} - \overline{S}) = \overline{K}_2(1 - \overline{K}_2)\overline{B} + 2\overline{\overset{(10)1}{B}}. \tag{3.6}$$

The relation (3.5) follows by substituting (3.3), (3.4), (3.2)b, (2.16)c, into (3.6). □

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