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ON SOME FUZZY QUOTIENT GROUPS

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I. Introduction.

Let G be a group and μ a fuzzy subgroup of G. For any $x \in G$, consider a map $\hat{\mu}_x : G \to [0, 1]$ defined by $\hat{\mu}_x(g) = \mu(gx^{-1})$ for all $g \in G$. In this case, $\hat{\mu}_x$ is called the fuzzy coset of G determined by x and μ. We put $K = \{x \in G | \mu(x) = \mu(e)\}$, where e is the identity of G, and N denotes a normal subgroup of G. Suppose $\Im$ is the set of all the fuzzy cosets of G by μ, and define a map. $\bar{\mu} : \Im \to [0, 1]$ by $\bar{\mu}(\hat{\mu}_x) = \sup_{n \in N} \hat{\mu}_x(n)$.

The concepts of fuzzy subsets was introduced by L.A. Zadeh [9], after then fuzzy subgroups were first defined by A. Rosenfeld [8]. P.S. Das [5] studied level subgroups. The basic notions, some results of fuzzy cosets and fuzzy quotient groups were first studied by N.P. Mukherjee and P. Bhattacharya [7].

In this paper, by using the properties of fuzzy(normal) subgroups, fuzzy cosets and basic group theory, we will investigate another kind of fuzzy quotient groups $\hat{\mu}$, $\bar{\mu}$ defined by

$$\hat{\mu}(Kx) = \sup_{k \in K} \mu(kx) \ \forall x \in G \ \text{(Theorem 3.1)} ,$$
$$\bar{\mu}(\hat{\mu}_x) = \sup_{n \in N} \hat{\mu}_x(n) \ \forall x \in G \ \text{(Theorem 3.3)}$$

respectively.

II. Preliminaries and Some Basic Results.

We review some basic definitions and results. For details, see P.S. Das [5], N.P.Mukherjee and P. Bhattacharya [3,7], A. Rosenfeld [8] and L.A. Zadeh [9].

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DEFINITION 2.1. Let G be a set. A mapping $\mu : G \to [0, 1]$ is called a *fuzzy subset* of G.

DEFINITION 2.2. If μ is a fuzzy subset of a set G, then for any $t \in [0, 1]$, the set

$$\mu_t = \{x \in G | \mu(x) \geqq t\}$$

is called a *level subset* of μ.

DEFINITION 2.3. Let G be a group. A mapping $\mu : G \to [0, 1]$ is called a *fuzzy subgroup* of G if

(1) $$\mu(xy) \geqq min\{\mu(x), \mu(y)\} \ \forall x, y \in G,$$

(2) $$\mu(x^{-1}) = \mu(x) \ \forall x \in G.$$

It is easy to see that if μ is a fuzzy subgroup of a group G whose identity is denoted by e, then we have $\mu(x) \leqq \mu(e)$ for all $x \in G$.

If μ is a fuzzy subgroup of G, then for any $t \in [0\,1]$ with $t \leqq \mu(e)$, the level subset μ_t is a subgroup of G in the usual sense. In this situation, the level set μ_t is called a *level subgroup* of μ.

$N \lhd G$ denotes that N is a normal subgroup of the group G.

LEMMA 2.4 [7]. *Let μ be a fuzzy subgroup of a group G. Let $x \in G$. Then*

$$\mu(xy) = \mu(y) \ \forall y \in G \Longleftrightarrow \mu(x) = \mu(e).$$

Proof. Suppose that $\mu(xy) = \mu(y) \ \forall y \in G$. Then, by choosing $y = e$, we get $\mu(x) = \mu(e)$.

Conversely, suppose that $\mu(x) = \mu(e)$. Then, since $\mu(y) \leqq \mu(e)$ for all $y \in G$, we have $\mu(y) \leqq \mu(x)$.

Now $\mu(xy) \geqq min\{\mu(x), \mu(y)\}$. Therefore, we have

$$\mu(xy) \geqq \mu(y) \ \forall y \in G.$$

But $\mu(y) = \mu(x^{-1}xy) \geqq min\{\mu(x), \mu(xy)\}$. Since $\mu(x) \geqq \mu(xy) \ \forall y \in G$, the following holds:

$$min\{\mu(x), \mu(xy)\} = \mu(xy) \leqq \mu(y).$$

Therefore, we get

$$\mu(y) \geqq \mu(xy) \ \forall y \in G.$$

Hence the result follows. □

DEFINITION 2.5. A fuzzy subgroup μ of a group G is called a *fuzzy normal subgroup* of G if $\mu(xy) = \mu(yx)\ \forall x,\ y \in G$.

THEOREM 2.6 [7]. *A fuzzy subgroup μ of a group G is a fuzzy normal subgroup if and only if μ is constant on the conjugate classes of G.*

Proof. Suppose that μ is a fuzzy normal subgroup of G. Then

$$\mu(y^{-1}xy) = \mu(xyy^{-1}) = \mu(x)\ \forall x,\ y \in G.$$

Conversely, suppose that μ is constant on each conjugate class of G. Then

$$\mu(xy) = \mu(xyxx^{-1}) = \mu(x(yx)x^{-1}) = \mu(yx)\ \forall x,\ y \in G.$$

Hence, μ is a fuzzy normal subgroup of G. □

THEOREM 2.7 [1]. *Suppose that μ is a fuzzy normal subgroup of a group G. Let $t \in [0, 1]$ such that $t \leqq \mu(e)$, where e denotes the identity of G. Then the set*

$$\mu_t = \{x \in G | \mu(x) \geqq t\}$$

is a normal subgroup of G.

Proof. We have already mentioned that μ_t is a subgroup of G in usual sense(P.S. Das [5]). We now show that μ_t is a normal subgroup. Let $x \in \mu_t$ and $y \in G$. Since μ is a fuzzy normal subgroup, we have by Theorem 2.6 that $\mu(y^{-1}xy) = \mu(x)$. So, we get that $\mu(y^{-1}xy) \geqq t$, implying that $y^{-1}xy \in \mu_t$. Therefore, $y^{-1}xy \in \mu_t\ \forall x \in \mu_t$ and $y \in G$. Hence $\mu_t \lhd G$. □

III. Fuzzy Quotient Groups.

In this section, we treat main results (Theorem 3.1, Proposition 3.2, Theorem 3.4).

THEOREM 3.1. *Suppose μ is a fuzzy normal subgroup of a group G with the identity e. Let*

$$K = \{x \in G | \mu(x) = \mu(e)\}.$$

Then $K \lhd G$. Consider a map $\hat{\mu} : G/K \to [0, 1]$ defined by

$$\hat{\mu}(Kx) = \sup_{k \in K} \mu(kx) \ \forall x \in G.$$

Then $\hat{\mu}$ is well-defined and $\hat{\mu}$ is a fuzzy subgroup of G/K. In this case, $\hat{\mu}$ is called the fuzzy quotient group of μ by K.

Proof. Since μ is a fuzzy normal subgroup, it follows from Theorem 2.7 that $K \lhd G$. Further, if $Kx = Ky$ for some $x, y \in G$, then $xy^{-1} \in K$ and so $\mu(xy^{-1}) = \mu(e)$. By Lemma 2.4, this give us that $\mu(kx) = \mu(ky)$ for $k \in K$, that is, $\hat{\mu}(Kx) = \hat{\mu}(Ky)$. Therefore, $\hat{\mu}$ is a well-defined map.

It is easy to check the followings:

$$\begin{aligned}
\hat{\mu}(KxKy) &= \hat{\mu}(Kxy) = \sup_{k \in K} \mu(kxy) \\
&\geqq \sup_{k_1, k_2 \in K} \min\{\mu(k_1 x), \mu(k_2 y)\} \\
&\geqq \min\{\sup_{k_1 \in K} \mu(k_1 x), \sup_{k_2 \in K} \mu(k_2 y)\} \\
&= \min\{\hat{\mu}(Kx), \hat{\mu}(Ky)\}, \\
\hat{\mu}((Kx)^{-1}) &= \hat{\mu}(Kx^{-1}) = \sup_{k \in K} \mu(kx^{-1}) \\
&= \sup_{k \in K} \mu(xk) = \sup_{k \in K} \mu(kx) = \hat{\mu}(Kx).
\end{aligned}$$

Hence $\hat{\mu}$ is a fuzzy subgroup of G/K.

However, $\hat{\mu}$ is not fuzzy normal, since

$$\hat{\mu}(KxKy) \neq \hat{\mu}(KyKx). \quad \square$$

REMARK. It is easy to see that if we define as $\hat{\mu}(Kx) = \mu(x) \ \forall x \in G$ in the above Theorem 3.1, then $\hat{\mu}$ is a fuzzy normal subgroup of G/K.

PROPOSITION 3.2. *Suppose that $f : G \to G'$ is an onto group homomorphism with kernel K, and let μ be a fuzzy subgroup of G. Then, for each $t \in [0, 1)$*

$$(\hat{\mu})_t = K\mu_t/K.$$

Proof. For $Kx \in (\hat{\mu})_t$, it holds that if $\hat{\mu}(Kx) \geqq t$ for all $x \in G$ then $\sup_{k\in K} \mu(kx) \geqq t$. So that, $\mu(k_0x) \geqq t$ for some $k_0 \in K$. This implies $k_0x \in \mu_t$, and hence $k_0x \in K\mu_t$. Therefore $Kk_0x = Kx \in K\mu_t/K$. Consequently, $(\hat{\mu})_t \subseteq K\mu_t/K$.

For the reverse inclusion, let $Kx \in K\mu_t/K$. Then $Kx = Kx_0$ for some $x_0 \in \mu_t$. So that

$$\hat{\mu}(Kx) = \hat{\mu}(Kx_0) = \sup_{k\in K} \mu(kx_0) \geqq t.$$

Therefore, $Kx \in (\hat{\mu})_t$ and hence $K\mu_t/K \subseteq (\hat{\mu})_t$. □

DEFINITION 3.3 [7]. Let μ be a fuzzy subgroup of a group G. For any $x \in G$, define a map

$$\hat{\mu}_x : G \to [0, 1]$$

by

$$(1) \qquad \hat{\mu}_x(g) = \mu(gx^{-1}) \ \forall g \in G.$$

In this case, $\hat{\mu}_x$ is called the *fuzzy coset* of G determined by x and μ.

THEOREM 3.4. *Let N be a normal subgroup of a group G. Suppose that μ is a fuzzy normal subgroup of G. Let $\Im$ be the set of all the fuzzy cosets of G by μ. Then $\Im$ is a group under the composition defined by*

$$(2) \qquad \hat{\mu}_x \circ \hat{\mu}_y = \hat{\mu}_{xy} \ \forall x, y \in G.$$

Define a map $\bar{\mu} : \Im \to [0, 1]$ by

$$(3) \qquad \bar{\mu}(\hat{\mu}_x) = \sup_{n\in N} \hat{\mu}_x(n) = \sup_{n\in N} \mu(nx^{-1}) \ \forall x \in G.$$

Then $\bar{\mu}$ is a fuzzy subgroup of $\Im$. In this case, $\bar{\mu}$ is called the fuzzy quotient group determined by μ and N.

Proof. First, we show that the composition(2) is well-defined. Let $x,\ y,\ x_0,\ y_0 \in G$ such that

$$\hat{\mu}_x = \hat{\mu}_{x_0} \text{ and } \hat{\mu}_y = \hat{\mu}_{y_0}. \tag{4}$$

Then we must show that

$$\hat{\mu}_x \circ \hat{\mu}_y = \hat{\mu}_{x_0} \circ \hat{\mu}_{y_0},$$

that is, $\hat{\mu}_{xy} = \hat{\mu}_{x_0 y_0}$.

We have, by definition,

$$\hat{\mu}_{xy}(g) = \mu(gy^{-1}x^{-1}) \ \forall g \in G,$$
$$\hat{\mu}_{x_0 y_0}(g) = \mu(gy_0^{-1}x_0^{-1}) \ \forall g \in G.$$

Now,

(5)
$$\mu(gy^{-1}x^{-1}) = \mu(gy_0^{-1}y_0y^{-1}x^{-1}) = \mu(gy_0^{-1}x_0^{-1}x_0y_0y^{-1}x^{-1})$$
$$\geqq min\,\{\mu(gy_0^{-1}x_0^{-1}),\ \mu(x_0y_0y^{-1}x^{-1})\}.$$

Again, from (4) we have

$$\mu(gx^{-1}) = \mu(gx_0^{-1}) \ \forall g \in G, \tag{6}$$
$$\mu(gy^{-1}) = \mu(gy_0^{-1}) \ \forall g \in G. \tag{7}$$

Now, in (6), substituting $x_0y_0y^{-1}$ for g, we have

$$\begin{aligned}\mu(x_0y_0y^{-1}x^{-1}) &= \mu(x_0y_0y^{-1}x_0^{-1})\\ &= \mu(y_0y^{-1}) \quad \text{(since } \mu \text{ is fuzzy normal)}\\ &= \mu(e) \quad \text{(by Lemma 2.4)}\ .\end{aligned}$$

But $\mu(e) \geqq \mu(gy_0^{-1}x_0^{-1})$, since for any fuzzy group μ, $\mu(e) \geqq \mu(x)\ \forall x \in G$. Thus, from (5) we get

$$\mu(gy^{-1}x^{-1}) \geqq \mu(gy_0^{-1}x_0^{-1}).$$

Similarly, substituting xyy_0^{-1} for g in (6) and using μ being fuzzy normal, it follows that

$$\mu(xyy_0^{-1}x_0^{-1}) = \mu(xyy_0^{-1}x^{-1}) = \mu(yy_0^{-1}) = \mu(e).$$

So, we have

$$\begin{aligned}\mu(gy_0^{-1}x_0^{-1}) &= \mu(gy^{-1}yy_0^{-1}x_0^{-1}) = \mu(gy^{-1}x^{-1}xyy_0^{-1}x_0^{-1}) \\ &\geqq min\{\mu(gy^{-1}x^{-1}),\ \mu(xyy_0^{-1}x_0^{-1})\} \\ &= \mu(gy^{-1}x^{-1}).\end{aligned}$$

Hence, we have $\mu(gy_0^{-1}x_0^{-1}) = \mu(gy^{-1}x^{-1})$, that is $\hat{\mu}_{x_0y_0} = \hat{\mu}_{xy}$, and therefore we have established that the composition (2) is well-defined. The composition defined in (2) is clearly associative. Since $\hat{\mu}_x \circ \hat{\mu}_{x^{-1}} = \hat{\mu}_{x^{-1}} \circ \hat{\mu}_x = \hat{\mu}_e$ for $x \in G$, we have that the inverse of $\hat{\mu}_x$ is $\hat{\mu}_{x^{-1}}$ for $x \in G$. Hence it follows that $\Im$ is a group.

Now, let $x,\ y \in G$. Then we have that

$$\begin{aligned}\bar{\mu}(\hat{\mu}_x \circ \hat{\mu}_y) &= \bar{\mu}(\hat{\mu}_{xy}) = \sup_{n\in N} \hat{\mu}_{xy}(n) \\ &= \sup_{n\in N} \mu(ny^{-1}x^{-1}) \\ &\geqq \sup_{n_1,n_2\in N} min\{\mu(n_1y^{-1}),\ \mu(n_2x^{-1})\} \\ &\geqq min\{\sup_{n_1\in N} \mu(n_1y^{-1}),\ \sup_{n_2\in N} \mu(n_2x^{-1})\} \\ &= min\{\sup_{n_1\in N} \hat{\mu}_y(n_1),\ \sup_{n_2\in N} \hat{\mu}_x(n_2)\} \\ &= min\{\bar{\mu}(\hat{\mu}_y),\ \bar{\mu}(\hat{\mu}_x)\}.\end{aligned}$$

Further, we have

$$\begin{aligned}\bar{\mu}(\hat{\mu}_x^{-1}) &= \bar{\mu}(\hat{\mu}_{x^{-1}}) = \sup_{n\in N} \hat{\mu}_{x^{-1}}(n) = \sup_{n\in N} \mu(nx) \\ &= \sup_{n\in N} \mu(x^{-1}n) = \sup_{n\in N} \mu(nx^{-1}) \\ &= \sup_{n\in N} \hat{\mu}_x(n) = \bar{\mu}(\hat{\mu}_x).\end{aligned}$$

Hence, it follows that $\bar{\mu}$ is a fuzzy subgroup of $\Im$.

However, $\bar{\mu}$ is not fuzzy normal, since $\bar{\mu}(\hat{\mu}_x \circ \hat{\mu}_y) \neq \bar{\mu}(\hat{\mu}_y \circ \hat{\mu}_x)$. □

REMARK. It is easy to see that if we define as $\bar{\mu}(\hat{\mu}_x) = \mu(x)\ \forall x \in G$ in the above Theorem 3.4, then $\bar{\mu}$ is a fuzzy normal subgroup of $\Im$.

COROLLARY 3.5. *With the same notations as in Definition 3.3 and Theorem 3.4, consider a map*

$$\theta : G \to \Im \text{ defined by } \theta(x) = \hat{\mu}_x.$$

Then θ is a homomorphism with kernel given by

$$K = \{x \in G | \mu(x) = \mu(e)\},$$

where e is the identity of G.

Proof. Let $x,\ y \in G$. Then

$$\theta(xy) = \hat{\mu}_{xy} = \hat{\mu}_x \circ \hat{\mu}_y = \theta(x) \circ \theta(y).$$

Hence θ is a homomorphism.

Further, the kernel K of θ is as follows:

$$\begin{aligned} K &= \{x \in G | \theta(x) = \hat{\mu}_e\} = \{x \in G | \hat{\mu}_x = \hat{\mu}_e\} \\ &= \{x \in G | \mu(yx^{-1}) = \mu(y) \text{ for all } y \in G\} \\ &= \{x \in G | \mu(x^{-1}) = \mu(e)\} \\ &= \{x \in G | \mu(x) = \mu(e)\}. \quad \square \end{aligned}$$

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