

**SOME RESULTS OF OPERATORS
IN THE CLASS $\mathbb{A}_{m,n}^l$**

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I. INTRODUCTION

Let \mathcal{H} be a separable, infinite dimensional, complex Hilbert spaces and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on \mathcal{H} . A *dual algebra* is a subalgebra of $\mathcal{L}(\mathcal{H})$ that contains the identity operator $1_{\mathcal{H}}$ and is closed in the ultraweak operator topology on $\mathcal{L}(\mathcal{H})$. For $T \in \mathcal{L}(\mathcal{H})$, let \mathcal{A}_T denote the smallest subalgebra of $\mathcal{L}(\mathcal{H})$ that contains T and $1_{\mathcal{H}}$ and is closed in the ultraweak operator topology. Moreover, let $Q_{\mathcal{A}_T}$ denote the quotient space $\mathcal{C}_1(\mathcal{H})/\perp_{\mathcal{A}_T}$, where $\mathcal{C}_1(\mathcal{H})$ is the trace class ideal in $\mathcal{L}(\mathcal{H})$ under the trace norm, and $\perp_{\mathcal{A}_T}$ denotes the preannihilator of \mathcal{A}_T in $\mathcal{C}_1(\mathcal{H})$. For a brief notation, we shall denote $Q_{\mathcal{A}_T}$ by Q_T . One knows that \mathcal{A}_T is the dual space of Q_T and that the duality is given by

$$(1) \quad \langle A, [L] \rangle = tr(AL), \quad A \in \mathcal{A}_T, [L] \in Q_T.$$

The Banach space Q_T is called a predual of \mathcal{A}_T . For x and y in \mathcal{H} , we can write $x \otimes y$ for the rank one operator in $\mathcal{C}_1(\mathcal{H})$ defined by

$$(2) \quad (x \otimes y)(u) = (u, y)x \quad \text{for all } u \in \mathcal{H}.$$

The theory of dual algebras is applied to the study of invariant subspaces, dilation theory, and reflexivity. The classes $\mathbb{A}_{m,n}$ (to be defined in section 2) were defined by Bercovici-Foias-Pearcy in [2]. Also these classes are closely related to the study of the theory of dual algebras. In 1991, Chevreau-Pearcy [7] studied for the first time common noncyclic

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vectors for the families of operators in order to solve the invariant subspace problem of bounded operators whose spectral radius is one. As a sequel to this study, in this paper we consider some dilation theories for the membership in these classes $\mathbb{A}_{m,n}^l$ (to be defined in section 3).

2. NOTATION AND PRELIMINARIES

Recall that any contraction T can be written as a direct sum $T = T_1 \oplus T_2$, where T_1 is a completely nonunitary contraction and T_2 is a unitary operator. If T_2 is absolutely continuous or acts on the space (0) , T will be called an *absolutely continuous contraction*. We denote by $\mathbb{A} = \mathbb{A}(\mathcal{H})$ the class of all absolutely continuous contractions T in $\mathcal{L}(\mathcal{H})$ for which the Foias-Sz.Nagy functional calculus $\Phi_T: \mathcal{H}^\infty \rightarrow \mathcal{A}_T$ is an isometry. ([3], Theorem 4.1)

DEFINITION 2.1. ([10]) Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be a dual algebra and let m and n be any cardinal numbers such that $1 \leq m, n \leq \aleph_0$. A dual algebra \mathcal{A} will be said to have property $(\mathbb{A}_{m,n})$ if $m \times n$ system of simultaneous equations of the form

$$(3) \quad [x_i \otimes y_j] = [L_{ij}], \quad 0 \leq i < m, \quad 0 \leq j < n,$$

where $\{[L_{ij}]\}_{\substack{0 \leq i < m \\ 0 \leq j < n}}$ is an arbitrary $m \times n$ array from $Q_{\mathcal{A}}$, has a solution $\{x_i\}_{0 \leq i < m}, \{y_j\}_{0 \leq j < n}$ consisting of a pair of sequences of vectors from \mathcal{H} . Furthermore, if m and n are positive integers and r is a fixed real number satisfying $r \geq 1$, a dual algebra \mathcal{A} (with property $(\mathbb{A}_{m,n})$) is said to have property $(\mathbb{A}_{m,n}(r))$ if for every $s > r$ and every $m \times n$ array $\{[L_{i,j}]\}_{\substack{0 \leq i < m \\ 0 \leq j < n}}$ from $Q_{\mathcal{A}}$, there exist sequences $\{x_i\}_{0 \leq i < m}$ and $\{y_j\}_{0 \leq j < n}$ from \mathcal{H} that satisfy (3) and also satisfy the following conditions:

$$(4a) \quad \|x_i\|^2 \leq s \sum_{0 \leq j < n} \|[L_{ij}]\|, \quad 0 \leq i < m,$$

and

$$(4b) \quad \|y_j\|^2 \leq s \sum_{0 \leq i < m} \|[L_{ij}]\|, \quad 0 \leq j < n.$$

For brief notation, we shall denote $(\mathbb{A}_{n,n})$ by (\mathbb{A}_n) . Furthermore, if m and n are cardinal numbers such that $1 \leq m, n \leq \aleph_0$, we denote by $\mathbb{A}_{m,n} = \mathbb{A}_{m,n}(\mathcal{H})$ the set of all T in $\mathbb{A}(\mathcal{H})$ such that the singly generated dual algebra \mathcal{A}_T has property $(\mathbb{A}_{m,n})$.

DEFINITION 2.2. ([13]) Let \mathbb{N} be the set of all natural numbers. If $n \in \mathbb{N}$, we denote by $\tilde{\mathcal{H}}^{(n)}$ the Hilbert space consisting of the direct sum of n copies of \mathcal{H} and by $T^{(n)}$ the n -fold *ampliation* of T acting on $\tilde{\mathcal{H}}^{(n)}$ defined by

$$(5) \quad T^{(n)}(x_1 \oplus \cdots \oplus x_n) = Tx_1 \oplus \cdots \oplus Tx_n.$$

Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ is a dual algebra. We define

$$\mathcal{A}^{(n)} = \overbrace{\mathcal{A} \oplus \cdots \oplus \mathcal{A}}^{n \text{ copies}}, \mathcal{A} \in \mathcal{A}.$$

Then $\mathcal{A}^{(n)}$ is indeed a dual algebra on $\tilde{\mathcal{H}}^{(n)}$. For each T in $\mathcal{L}(\mathcal{H})$ it is clear that $(\mathcal{A}_T)^{(n)} = \mathcal{A}_{T^{(n)}}$. We shall employ the notation $C_0 = C_0(\mathcal{H})$ for the class of all (completely nonunitary) contractions T in $\mathcal{L}(\mathcal{H})$ such that the sequences $\{T^{*n}\}$ converges to zero in the strong operator topology and is denoted by, as usual, $C_0 = (C_0)^*$.

3. MAIN RESULTS

DEFINITION 3.1. ([9]) Let m, n and l be any cardinal numbers such that $1 \leq m, n, l \leq \aleph_0$. We denote by $\mathbb{A}_{m,n}^l(\mathcal{H})$ the class of all $\{T_k\}_{k=1}^l$ in $\mathbb{A}(\mathcal{H})$ such that every $m \times n \times l$ system of simultaneous equations of the form

$$(6) \quad [x_i \otimes y_j^{(k)}]_{T_k} = [L_{ij}^{(k)}]_{T_k},$$

where $\{[L_{ij}^{(k)}]_{T_k}\}_{\substack{0 \leq i < m \\ 0 \leq j < n}}$ is an arbitrary $m \times n$ array from Q_{T_k} for each $k = 1, \dots, l$, has a solution $\{x_i\}_{0 \leq i < m}, \{y_j^{(k)}\}_{\substack{0 \leq j < n \\ 1 \leq k \leq l}}$ consisting of a pair

of sequences of vectors from \mathcal{H} . Furthermore, if for every doubly indexed family $\{[L_{ij}^{(k)}]_{T_k}\}_{\substack{0 \leq i < m \\ 0 \leq j < n}}$ of elements of Q_{T_k} for each $k = 1, \dots, l$,

such that the rows and columns of the matrix $(\|[L_{ij}^{(k)}]_{T_k}\|)$ are summable and r is a fixed real number satisfying $r \geq 1$, then we denote by $\mathbb{A}_{m,n}^l(r)$ the class of all $\{T_k\}_{k=1}^l$ in $\mathbb{A}(\mathcal{H})$ such that every $m \times n \times l$ system of simultaneous equations of the form (6) has a solution $\{x_i\}_{0 \leq i < m}, \{y_j^{(k)}\}_{\substack{0 \leq j < n \\ 1 \leq k \leq l}}$ consisting of a pair of sequences of vectors from \mathcal{H} and also satisfy the following conditions, for every $s > r$,

$$(7) \quad \|x_i\|^2 \leq s \sum_{0 \leq j < n} \|[L_{ij}^{(k)}]_{T_k}\|, \quad 0 \leq i < m, 1 \leq k \leq l$$

and

$$(8) \quad \|y_j^{(k)}\|^2 \leq s \sum_{0 \leq i < m} \|[L_{ij}^{(k)}]_{T_k}\|, \quad 0 \leq j < n, 1 \leq k \leq l.$$

LEMMA 3.2. ([12]) Suppose $T, S \in \mathbb{A}(\mathcal{H})$. Let $[L]_T \in Q_T$ and let $[M]_S \in Q_S$. Then

$$\phi_T([L]_T) = \phi_S([M]_S)$$

if and only if

$$\langle T^n, [L]_T \rangle = \langle S^n, [M]_S \rangle, n = 0, 1, 2, \dots.$$

We are ready to prove main results. Suppose \mathcal{H}_r is a separable infinite dimensional complex Hilbert space for $r = 1, 2, \dots, m$.

THEOREM 3.3. Assume $\{T_k^{(r)}\}_{k=1}^m \in \mathbb{A}_{1,1}^m(\mathcal{H}_r)$ for each $r = 1, 2, \dots, m$. Let $\mathcal{H} = \bigoplus_{r=1}^m \mathcal{H}_r$ and $T_k = \bigoplus_{r=1}^m T_k^{(r)}$. Then $\{T_k\}_{k=1}^m \in \mathbb{A}_{1,m}^m(\mathcal{H})$.

Proof. We first show that $T_k \in \mathbb{A}(\mathcal{H})$ for all k . If $f \in \mathbb{H}^\infty(\mathbb{D})$, then $\|f(T_k)\| = \sup_r \|f(T_k^{(r)})\| = \|f\|_\infty$, since $T_k^{(r)} \in \mathbb{A}(\mathcal{H}_r)$ for $k = 1, 2,$

\dots, m . To show that $\{T_k\}_{k=1}^m \in \mathbb{A}_{1,m}^m(\mathcal{H})$, suppose we are given array $\{[L_r^{(k)}]_{T_k}\}_{1 \leq r \leq m}$ in Q_{T_k} for each $k = 1, \dots, m$. Let us consider the usual isometric isomorphic weak* homeomorphism $\phi_{k,r}$ from Q_{T_k} onto $Q_{T_k^{(r)}}$. Since $\phi_{k,r}([L_r^{(k)}]_{T_k}) \in Q_{T_k^{(r)}}$, by definition, for each r , there exist vectors $x^{(r)}$ and $\{y_k^{(r)}\}_{k=1}^m$ in \mathcal{H}_r such that

$$(9) \quad \phi_{k,r}([L_r^{(k)}]_{T_k}) = [x^{(r)} \otimes y_k^{(r)}]_{T_k^{(r)}}, \quad 1 \leq k \leq m.$$

Letting $\tilde{x} = \bigoplus_{r=1}^m x^{(r)}$ and let $\tilde{w}_r^{(k)}$ be the vector in \mathcal{H} that has $y_k^{(r)}$ as r -component and zeros elsewhere, for $1 \leq r, k \leq m$. Then $\tilde{x}, \tilde{w}_r^{(k)} \in \mathcal{H}$ and we have, by lemma 3.2 and (9),

$$(10) \quad \begin{aligned} \phi_{k,r}([L_r^{(k)}]_{T_k}) &= [x^{(r)} \otimes y_k^{(r)}]_{T_k^{(r)}}, \quad 1 \leq k, r \leq m \\ &= \phi_{k,r}([\tilde{x} \otimes \tilde{w}_r^{(k)}]_{T_k}). \end{aligned}$$

So we get

$$(11) \quad [L_r^{(k)}]_{T_k} = [\tilde{x} \otimes \tilde{w}_r^{(k)}]_{T_k}$$

for $1 \leq r \leq m$ and $1 \leq k \leq m$. Therefore, we have $\{T_k\}_{k=1}^m \in \mathbb{A}_{1,m}^m(\mathcal{H})$.

THEOREM 3.4. Assume $\{T_k^{(r)}\}_{k=1}^m \in \mathbb{A}_{1,n}^m(\mathcal{H}_r)$ for each $r = 1, 2, \dots, m$. Let $\mathcal{H} = \bigoplus_{r=1}^m \mathcal{H}_r$ and $T_k = \bigoplus_{r=1}^m T_k^{(r)}$. Then $\{T_k\}_{k=1}^m \in \mathbb{A}_{m,n}^m(\mathcal{H})$.

Proof. The proof is similar to that of Theorem 3.3.

THEOREM 3.5. Suppose \mathcal{H}_w is a separable infinite dimensional complex Hilbert space for $w = 1, \dots, m$. Suppose also that $\{T_k^{(w)}\}_{k=1}^m \in \mathbb{A}_{1,n_w}^m(\mathcal{H}_w)(r_w)$ where $n_w \in \mathbb{N}$ and $r_w \geq 1$ for all $w, 1 \leq w \leq m$. Let $\mathcal{H} = \bigoplus_{w=1}^m \mathcal{H}_w$ and $T_k = \bigoplus_{w=1}^m T_k^{(w)}$. Then, we have

$$\{T_k\}_{k=1}^m \in \mathbb{A}_{m,N}^m(\mathcal{H})(r)$$

where $N = m(n_1 + \cdots + n_m)$ and $r = \max\{r_w : 1 \leq w \leq m\}$.

Proof. By theorem 3.3, $T_k \in \mathbb{A}(\mathcal{H})$ for all k . To show that $\{T_k\}_{k=1}^m \in \mathbb{A}_{m,N}^m(\mathcal{H})(r)$, let $s > r$ and suppose we are given array $\{[L_{wj}^{(k)}]_{T_k}\}_{\substack{1 \leq w \leq m \\ 1 \leq j \leq N}}$ in Q_{T_k} for each $k = 1, \dots, m$. Let us denote $\phi_{k,w} = \phi_{T_k}^{-1} \circ \phi_{T_k}$. Then it is obvious that $\phi_{k,w}$ is an isometric isomorphic weak* homeomorphism from Q_{T_k} onto $Q_{T_k^{(w)}}$. Since $\phi_{k,w}([L_{wj}^{(k)}]_{T_k}) \in Q_{T_k^{(w)}}$, for any $s > r_w$, there exist a vector $x^{(w)}$ and sequence of vectors $\{y_j^{(k,w)}\}_{j=1, k=1}^{n_w, m}$ in \mathcal{H}_w , $1 \leq w \leq m$ such that

$$(12) \quad \phi_{k,w}([L_{wj}^{(k)}]_{T_k}) = [x^{(w)} \otimes y_j^{(k,w)}]_{T_k^{(w)}}$$

for j and k , $1 \leq j \leq n_w$, $1 \leq w \leq m$ and furthermore,

$$(13) \quad \|x^{(w)}\|^2 \leq s \sum_{j=1}^{n_w} \|[L_{wj}^{(k)}]_{T_k}\|, \quad 1 \leq k \leq m,$$

and

$$(14) \quad \|y_j^{(k,w)}\|^2 \leq s \sum_{w=1}^m \|[L_{wj}^{(k)}]_{T_k}\|, \quad 1 \leq k \leq m.$$

Now let \tilde{x}_w be the vector in \mathcal{H} that has $x^{(w)}$ as w -component and zeros elsewhere and let $\tilde{v}_j^{(k)}$ be the vector in \mathcal{H} that has $y_j^{(k,w)}$ as w -component and zeros elsewhere, for $1 \leq j \leq n_w$, $1 \leq w, k \leq m$. Then it follows from Lemma 3.2 and (12) that we have

$$(15) \quad \begin{aligned} \phi_{k,w}([L_{wj}^{(k)}]_{T_k}) &= [x^{(w)} \otimes y_j^{(k,w)}]_{T_k^{(w)}} \\ &= \phi_{k,w}([\tilde{x}_w \otimes \tilde{v}_j^{(k)}]_{T_k}), \end{aligned}$$

and for $1 \leq j \leq n_w$, $1 \leq w, k \leq m$. So we get

$$(16) \quad [L_{wj}^{(k)}]_{T_k} = [\tilde{x}_w \otimes \tilde{v}_j^{(k)}]_{T_k}$$

for $1 \leq w, k \leq m, 1 \leq j \leq n_1 + \cdots + n_m$. Furthermore, we have

$$(17) \quad \|\tilde{x}_w\|^2 = \|x^{(w)}\|^2 \leq s \sum_{j=1}^{n_w} \|[L_{wj}^{(k)}]_{T_k}\| \\ \leq s \sum_{j=1}^N \|[L_{wj}^{(k)}]_{T_k}\|, \quad 1 \leq k \leq m,$$

and

$$(18) \quad \|\tilde{v}_j^{(k)}\|^2 = \|y_j^{(k,w)}\|^2 \leq s \sum_{w=1}^m \|[L_{wj}^{(k)}]_{T_k}\|, \quad 1 \leq k \leq m.$$

Therefore, we have

$$\{T_k\}_{k=1}^m \in \mathbb{A}_{m,N}^m(\mathcal{H})(r),$$

where $r = \max\{r_w : 1 \leq w \leq m\}$.

References

1. C. Apostol, H. Bercovici, C. Foias and C. Pearcy, *Invariant subspaces, dilation theory, and structure of the predual of a dual algebra I*, *Funct. Anal.* **63** (1985), 369–404.
2. H. Bercovici, B. Chevreau, C. Foias and C. Pearcy, *Dilation theory and systems of simultaneous equations in the predual of an operator algebra I*, *Michigan Math. J.* **30** (1983), 335–354.
3. ———, *Dual algebras with applications to invariant subspaces and dilation theory*, C.B.M.S. Regional Conference Series, no. 56, A.M.S., Providence, R. I.
4. A. Brown and C. Pearcy, *Introduction to operator theory I*, *Elements of functional analysis*, Springer-Verlag, New York, 1977.
5. B. Chevreau, G. Exner, and C. Pearcy, *On the structure of contraction operators III*, *Michigan Math. J.* **36** (1989), 29–62.
6. B. Chevreau, and C. Pearcy, *On the structure of contraction operators I*, *J. Funct. Anal.*, **76** (1988), 1–29.
7. ———, *On common noncyclic vectors for families of operators*, *Houston J. of Math.* vol. 17, No.4, (1991), 637–650.
8. G. Exner and P. Sullivan, *Normal operators and the classes \mathbb{A}_n* , *J. Operator Theory*, 19, (1988), 81–94.
9. Hae Gyu Kim, *Ph.D. Thesis*, Kyungpook National University, 1992.

10. I. Jung, *Dual operator algebras and the classes $\mathbb{A}_{m,n}$ I*, J. Operator Theory, 27, (1992), 309-323.
11. I. B. Jung and C. Y. Park, *A note on operators in the classes $\mathbb{A}_{m,n}$* , Kyungpook Math. J. **28**, No. 28 (1988), 173-176.
12. I. B. Jung and Y. C. Kim, *Dual operator algebras and C_0 -operators*, Bulletin of the Korean Mathematical Society . **27**, No. 2, (1990), 173-182.
13. B. Sz-Nagy and C. Foias, *Harmonic analysis of operators on Hilbert spaces*, North Holland Akademiai Kiado, Amsterdam/Budapest, 1970.

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