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THE g–RECURRENT CONDITION IMPOSED ON THE EINSTEIN'S CONNECTION

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1. Introduction.

Various recurrent connections have been studied by many authors, such as Chung, Datta, E.M. Patterson, M. Prvanovitch, Singal, and Takano, etc(refer to [4] and [5]). Examples of such connections are that of Ricci-recurrent curvature, that of birecurrent curvature, and skew-symmetric recurrent connection. In this paper, we introduce a new concept of g-recurrent connection in a generalized n-dimensional Riemannian manifold X_n, and prove that g-recurrent condition imposed on the Einstein's connection is meaningless from the physical point of view.

2. Preliminaries.

This section is a brief collection of definitions and notations which are needed in our subsequent considerations. Let X_n be a generalized n-dimensional Riemannian manifold referred to a real coordinate system x^ν , which obeys only coordinate transformations $x^\nu \to x^{\nu'}$ for which

$$Det(\frac{\partial x'}{\partial x}) \neq 0 \tag{2.1}$$

The manifold X_n is endowed with a general real nonsymmetric tensor $g_{\lambda\mu}$ which may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$[1]

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu} \tag{2.2}$$

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[1]Throughout the present paper, all Greek indices take the values $1, 2, \ldots, , n$ and follow the summation convention unless stated otherwise.

where

(2.3) $\mathfrak{g} = Det(g_{\lambda\mu}) \neq 0, \quad \mathfrak{h} = Det(h_{\lambda\mu}) \neq 0, \quad \mathfrak{k} = Det(k_{\lambda\mu})$

In virtue of (2.3) we may define a unique tensor $h^{\lambda\nu}$ by

(2.4) $$h_{\lambda\mu}h^{\lambda\nu} = \delta_\mu^\nu$$

which together with $h_{\lambda\mu}$ *will serve for raising and/or lowering indices of tensors in* X *in the usual manner.* There exists also a unique tensor ${}^*g^{\lambda\nu}$ satisfying

(2.5) $$g_{\lambda\mu}{}^*g^{\lambda\nu} = g_{\mu\lambda}{}^*g^{\nu\lambda} = \delta_\mu^\nu$$

The manifold X_n is connected by a general real connection $\Gamma_\lambda{}^\nu{}_\mu$ with the following transformation rule:

(2.6) $$\Gamma_{\lambda'}{}^{\nu'}{}_{\mu'} = \frac{\partial x^{\nu'}}{\partial x^\alpha}\left(\frac{\partial x^\beta}{\partial x^{\lambda'}}\frac{\partial x^\gamma}{\partial x^{\mu'}}\Gamma_\beta{}^\alpha{}_\gamma + \frac{\partial^2 x^\alpha}{\partial x^{\lambda'}\partial x^{\mu'}}\right)$$

It may also be decomposed into its symmetric part $\Lambda_\lambda{}^\nu{}_\mu$ and its skew-symmetric part $S_{\lambda\mu}{}^\nu$, called the torsion tensor of $\Gamma_\lambda{}^\nu{}_\mu$:

(2.7) $\Gamma_\lambda{}^\nu{}_\mu = \Lambda_\lambda{}^\nu{}_\mu + S_{\lambda\mu}{}^\nu; \quad \Lambda_\lambda{}^\nu{}_\mu = \Gamma_{(\lambda}{}^\nu{}_{\mu)}; \quad S_{\lambda\mu}{}^\nu = \Gamma_{[\lambda}{}^\nu{}_{\mu]}$

A connection $\Gamma_\lambda{}^\nu{}_\mu$ is said to be *einstein* if it satisfies the following system of Einstein's equations:

(2.8a) $$\partial_\omega g_{\lambda\mu} - \Gamma_\lambda{}^\alpha{}_\omega g_{\alpha\mu} - \Gamma_\omega{}^\alpha{}_\mu g_{\lambda\alpha} = 0$$

or equivalently

(2.8b) $$D_\omega g_{\lambda\mu} = 2S_{\omega\mu}{}^\alpha g_{\lambda\alpha}$$

where D_ω is the symbolic vector of the covariant derivative with respect to $\Gamma_\lambda{}^\nu{}_\mu$. The manifold X_n connected by this Einstein's connection is a generalization of the space-time X_4, and *Einstein's n-dimensional unified field theory* is based upon this manifold X_n. Our new concept of g-recurrent connection $\Gamma_\lambda{}^\nu{}_\mu$ is defined by the following system of equations:

(2.9) $$D_\omega g_{\lambda\mu} = 2X_\omega g_{\lambda\mu}$$

for a non-null vector X_μ. The manifold X_n connected by this connection is called an n-dimensional *g-recurrent manifold.*

The main purpose of the present paper is to prove that the Einstein's connection satisfying the g-recurrent condition (2.9) is meaningless from the physical point of view.

3. The g-recurrent connection.

This section is devoted to the investigations of the differential geometric properties of g-recurrent connections. The following two theorems will be proved simultaneously:

THEOREM 3.1. *The system (2.9) may be decomposed into*

$$D_\omega h_{\lambda\mu} = 2X_\omega h_{\lambda\mu} \tag{3.1a}$$

$$D_\omega k_{\lambda\mu} = 2X_\omega k_{\lambda\mu}. \tag{3.1b}$$

THEOREM 3.2. *The system (2.9) is equivalent to*

$$D_\omega {}^*g^{\lambda\nu} = -2X_\omega {}^*g^{\lambda\nu}. \tag{3.2}$$

Proof. The equations (3.1a,b) follow from (2.9) and

$$D_\omega h_{\lambda\mu} = D_\omega g_{(\lambda\mu)}, \quad D_\omega k_{\lambda\mu} = D_\omega g_{[\lambda\mu]}.$$

In virtue of (2.5), multiplication of ${}^*g^{\lambda\nu}$ to both sides of (2.9) gives

$$-g_{\lambda\mu} D_\omega {}^*g^{\lambda\nu} = {}^*g^{\lambda\nu} D_\omega g_{\lambda\mu} = 2X_\omega g_{\lambda\mu} {}^*g^{\lambda\nu} = 2X_\omega \delta_\mu^\nu \tag{3.3}$$

The equations (3.2) may be obtained by multiplying ${}^*g^{\epsilon\mu}$ again to both sides of (3.3). Conversely, start with (3.2), and multiply this equations by $g_{\lambda\mu}$ to get (2.9).

REMARK 3.3. *The form of equations (3.2) may be used for the study of g-recurrent connections in the Einstein's n-dimensional ${}^*g$-unified field theory(Refer to* [1],[2],[10]*).*

The following scalars will be used in our subsequent considerations:

$$g = \frac{\mathfrak{g}}{\mathfrak{h}}, \quad k = \frac{\mathfrak{k}}{\mathfrak{h}} \tag{3.4}$$

THEOREM 3.4. *The covariant derivative of the determinants* $\mathfrak{g}$ *and* $\mathfrak{h}$ *are*

$$D_\omega \mathfrak{g} = 2n\mathfrak{g}X_\omega \tag{3.5a}$$

$$D_\omega \mathfrak{h} = 2n\mathfrak{h}X_\omega. \tag{3.5b}$$

Proof. We first note that a direct consequence of (2.9) is

$$D_\omega \mathfrak{g} = \frac{\partial \mathfrak{g}}{\partial g_{\lambda\mu}} D_\omega g_{\lambda\mu} = \mathfrak{g}^* g^{\lambda\mu} D_\omega g_{\lambda\mu} \tag{3.6}$$

On the other hand, multiplication of $^*g^{\lambda\mu}$ to both sides of (2.9) gives

$$^*g^{\lambda\mu} D_\omega g_{\lambda\mu} = 2nX_\omega \tag{3.7}$$

The relation (3.5a) immediately follows by substituting (3.7) into (3.6). The relation (3.5b) may be proved similarly by starting from (2.4) and (3.1)a.

THEOREM 3.5. *If the system (2.9) admits a solution* $\Gamma_\lambda{}^\nu{}_\mu$, *it must be of the form*

$$\Gamma_\lambda{}^\nu{}_\mu = \{\lambda{}^\nu{}_\mu\} + S_{\lambda\mu}{}^\nu + V^\nu{}_{\lambda\mu} \tag{3.8}$$

where $\{\lambda{}^\nu{}_\mu\}$ *are the Christoffel symbols with respect to* $h_{\lambda\mu}$ *and*

$$V^\nu{}_{\lambda\mu} = V^\nu{}_{(\lambda\mu)} = -2S^\nu{}_{(\lambda\mu)} - 2X_{(\lambda}\delta_{\mu)}{}^\nu + X^\nu h_{\lambda\mu} \tag{3.9}$$

Proof. In virtue of

$$D_\omega h_{\lambda\mu} = \partial_\omega h_{\lambda\mu} - \Gamma_\lambda{}^\alpha{}_\omega h_{\alpha\mu} - \Gamma_\mu{}^\alpha{}_\omega h_{\lambda\alpha}$$

We have

$$\begin{aligned} &\frac{1}{2}h^{\nu\alpha}(D_\lambda h_{\alpha\mu} + D_\mu h_{\lambda\alpha} - D_\alpha h_{\lambda\mu}) \\ &= \{\lambda{}^\nu{}_\mu\} - 2h^{\nu\alpha}S_{\alpha(\lambda\mu)}\Gamma_{(\lambda}{}^\nu{}_{\mu)} \\ &= \{\lambda{}^\nu{}_\mu\} - 2S^\nu{}_{(\lambda\mu)} - \Gamma_\lambda{}^\nu{}_\mu + S_{\lambda\mu}{}^\nu \end{aligned} \tag{3.10}$$

On the other hand, the relation (3.1a) gives

$$\frac{1}{2}h^{\nu\alpha}(D_\lambda h_{\alpha\mu} + D_\mu h_{\lambda\alpha} - D_\alpha h_{\lambda\mu}) = 2X_{(\lambda}\delta_{\mu)}{}^\nu - X^\nu h_{\lambda\mu} \tag{3.11}$$

Comparing (3.10) and (3.11), we finally have (3.8) in virtue of (3.9).

REMARK 3.6. *In virtue of (3.8) and (3.9), we note that the investigation of the g-recurrent connection $\Gamma_\lambda{}^\nu{}_\mu$ is reduced to the study of the tensor $S_{\lambda\mu}{}^\nu$. In order to know the g-recurrent connection $\Gamma_\lambda{}^\nu{}_\mu$, it is necessary and sufficient to represent the tensor $S_{\lambda\mu}{}^\nu$ in terms of $g_{\lambda\mu}$. This is an open problem. Probably, the precise tensorial representation of $S_{\lambda\mu}{}^\nu$ in terms of $g_{\lambda\mu}$ may be obtained by starting from (3.1b).*

4. The g-recurrent condition imposed on the Einstein's connection.

In this section, we investigate the meaning of g-recurrent condition in the Einstein's n-dimensional unified field theory from physical point of view.

THEOREM 4.1. *A necessary condition for the system (2.9) to admit a solution is that the scalars g and k, defined by (3.4), are constant.*

Proof. In virtue of Theorem 3.1, we note that the system (2.9) is equivalent to (3.1a,b). Multiplication of ${}^*g^{\lambda\mu}$ to both sides of (2.9) gives

$$\begin{aligned}2nX_\omega &= (\partial_\omega g_{\lambda\mu} - \Gamma_\lambda{}^\alpha{}_\omega g_{\alpha\mu} - \Gamma_\mu{}^\alpha{}_\omega g_{\lambda\alpha})^* g^{\lambda\mu}\\ &= (\partial_\omega g_{\lambda\mu})^* g^{\lambda\mu} - 2\Gamma_\alpha{}^\alpha{}_\omega\\ &= \frac{1}{\mathfrak{g}}(\partial_\omega g_{\lambda\mu})\frac{\partial \mathfrak{g}}{\partial g_{\lambda\mu}} - 2\Gamma_\alpha{}^\alpha{}_\omega\\ &= \partial_\omega(\ln \mathfrak{g}) - 2\Gamma_\alpha{}^\alpha{}_\omega \qquad (4.1)\end{aligned}$$

Similarly, multiplying $h^{\lambda\mu}$ to both sides of (3.1a), we have

$$2nX_\omega = \partial_\omega(\ln \mathfrak{h}) - 2\Gamma_\alpha{}^\alpha{}_\omega \tag{4.2}$$

Comparing (4.1) and (4.2), we have

$$\partial_\omega(\ln \mathfrak{g}) = \partial_\omega(\ln \mathfrak{h}) \text{ or } g = \text{constant} \tag{4.3}$$

which proves the first statement. If $k = 0$, then our theorem is proved. If $k \neq 0$, then there exists a unique inverse tensor $\bar{k}^{\lambda\mu}$ such that

$$k_{\lambda\alpha}\bar{k}^{\nu\alpha} = \delta_\lambda^\nu \tag{4.4}$$

Consequently, multiplying $\bar{k}^{\lambda\mu}$ to both sides of (3.1b) it follows that

(4.5) $$2nX_\omega = \partial_\omega(\ln \mathfrak{k}) - 2\Gamma_\alpha{}^\alpha{}_\omega$$

which together with (4.2) give

$$k = \text{constant}.$$

REMARK 4.2. *In the Einstein's unified field theory, a function of scalar g may be identified with the gravitational function (Refer to [7],[9]). Therefore, if we assume that Einstein's connection is also g-recurrent in the Einstein's unified field theory, the gravitational function is reduced to a constant in the gravitational theory in virtue of Theorem 4.1. From the physical point of view, this is a strong restriction to the generality of Einstein's unified field theory. Consequently, the adoptation of the condition (2.9) in the Einstein's unified field theory is meaningless.*

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