

NONEXISTENCE OF POSITIVE SOLUTIONS

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1. Introduction.

The existence of positive solutions in a bounded domain of the following nonlinear elliptic equation has been extensively studied:

$$\begin{aligned} -\Delta u &= u^p + f(x, u) && \text{on } \Omega, \\ u &> 0 && \text{on } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

where $p = (n+2)/(n-2)$, $f(x, 0) = 0$ and $f(x, u)$ is a lower-order perturbation of u^p in the sense that $\lim_{u \rightarrow \infty} f(x, u)/u^p = 0$. See [BL], [AS], [L].

In this paper, we are concerned with the nonexistence of positive solutions of the equation

$$(1.1) \quad -\Delta u = \lambda u + u|u|^{2^*-2} \quad \text{in } \Omega,$$

$$(1.2) \quad u > 0 \quad \text{in } \Omega,$$

$$(1.3) \quad u = 0 \quad \text{on } \Omega,$$

where $2^* = 2n/(n-2)$.

2. Main Result

LEMMA 2.1. *Let V be a vector field on \mathbb{R}^n . Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be continuous with primitive $G(u) = \int_0^u g(v)dv$ and let $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ be a solution of the following equation*

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$$\begin{cases} -\Delta u = g(u) & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

in a domain $\Omega \subset \subset \mathbb{R}^n$. Then there holds

$$\begin{aligned} \frac{1}{2} \int_{\partial\Omega} \left| \frac{\partial u}{\partial \nu} \right|^2 \langle \nu, V \rangle d\sigma + \int_{\Omega} \left(\frac{|\nabla u|^2}{2} - G(u) \right) \operatorname{div} V dx \\ - \sum_{j=1}^n \int_{\Omega} \langle \nabla u, \nabla V_j \rangle \frac{\partial u}{\partial x_j} dx = 0, \end{aligned}$$

where ν denotes the exterior unit normal.

Proof.

$$\begin{aligned} \operatorname{div} (\nabla u \langle V, \nabla u \rangle) &= \Delta u \langle V, \nabla u \rangle + \sum_{i,j=1}^n \frac{\partial u}{\partial x_i} \frac{\partial V_j}{\partial x_i} \frac{\partial u}{\partial x_j} + \langle V, \nabla \left(\frac{|\nabla u|^2}{2} \right) \rangle \\ &= \nabla u \langle V, \nabla u \rangle + \sum_{j=1}^n \langle \nabla u, \nabla V_j \rangle \frac{\partial u}{\partial x_j} + \langle V, \nabla \left(\frac{|\nabla u|^2}{2} \right) \rangle. \end{aligned}$$

Hence

$$\nabla u \langle V, \nabla u \rangle = \operatorname{div} (\nabla u \langle V, \nabla u \rangle) - \sum_{j=1}^n \langle \nabla u, \nabla V_j \rangle \frac{\partial u}{\partial x_j} - \langle V, \nabla \left(\frac{|\nabla u|^2}{2} \right) \rangle.$$

Therefore

$$\begin{aligned} 0 &= (\Delta u + g(u)) \langle V, \nabla u \rangle \\ &= \Delta u \langle V, \nabla u \rangle + g(u) \langle V, \nabla u \rangle \\ &= \operatorname{div} (\nabla u \langle V, \nabla u \rangle) - \sum_{j=1}^n \langle \nabla u, \nabla V_j \rangle \frac{\partial u}{\partial x_j} - \langle V, \nabla \left(\frac{|\nabla u|^2}{2} \right) \rangle \\ &\quad + \langle V, \nabla G(u) \rangle \\ &= \operatorname{div} (\nabla u \langle V, \nabla u \rangle) - \frac{|\nabla u|^2}{2} \operatorname{div} V + G(u) \operatorname{div} V \\ &\quad + \frac{|\nabla u|^2}{2} \operatorname{div} V - G(u) \operatorname{div} V - \sum_{j=1}^n \langle \nabla u, \nabla V_j \rangle \frac{\partial u}{\partial x_j}. \end{aligned}$$

Observe that

$$\langle x, \nabla u \rangle = \langle x, \nu \rangle \frac{\partial u}{\partial \nu} \quad \text{and} \quad |\nabla u|^2 = \left| \frac{\partial u}{\partial \nu} \right|^2 \quad \text{on} \quad \partial \Omega.$$

Integrating this identity over Ω we have

$$\begin{aligned} \frac{1}{2} \int_{\partial \Omega} \left| \frac{\partial u}{\partial \nu} \right|^2 \langle \nu, V \rangle d\sigma + \int_{\Omega} \left(\frac{|\nabla u|^2}{2} - G(u) \right) \operatorname{div} V dx \\ - \sum_{j=1}^n \int_{\Omega} \langle \nabla u, \nabla V_j \rangle \frac{\partial u}{\partial x_j} dx = 0. \end{aligned}$$

We now apply this Lemma to obtain the following nonexistence result.

THEOREM 2.1. *Suppose $0 \in \Omega \neq \mathbb{R}^n$ is a smooth (possibly unbounded) domain in \mathbb{R}^n , $n \geq 3$. Suppose Ω has a vector field $V : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that*

- (1) $\langle \nu, V \rangle > 0$ on $\partial \Omega$,
- (2) $\operatorname{div} V = c$, c is a positive constant,
- (3) There is a positive constant M such that

$$\sum_{i,j}^n \frac{\partial V_j}{\partial x_j} \xi_i \xi_j \leq M |\xi|^2,$$

- (4) $nM \leq c$.

Then the boundary value problem (1.1), (1.2) and (1.3) has no positive solutions in H_0^1 .

Proof. Let $g(u) = \lambda u + u|u|^{2^*-2}$ with

$$G(u) = \frac{\lambda}{2} |u|^2 + \frac{1}{2^*} |u|^{2^*}.$$

We know that any solution of (1.1) and (1.3) is smooth on $\overline{\Omega}$. Hence by Lemma 2.1

$$\begin{aligned}
(2.1) \quad & \frac{1}{2} \int_{\partial\Omega} \left| \frac{\partial u}{\partial \nu} \right|^2 \langle \nu, V \rangle d\sigma + \frac{c}{2} \int_{\Omega} (|\nabla u|^2 - \lambda |u|^2 - \frac{2}{2^*} |u|^{2^*}) dx \\
& = \int_{\Omega} \left(\sum_{i,j=1}^n \frac{\partial V_j}{\partial x_i} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \right) dx.
\end{aligned}$$

Note that if u is a solution of (1.1), then

$$(2.2) \quad \int_{\Omega} (|\nabla u|^2 - \lambda |u|^2 - |u|^{2^*}) dx = 0.$$

Multiplying both sides of (2.1) by n/c and using the fact (2.2), we obtain

$$\begin{aligned}
& \frac{n}{2c} \int_{\partial\Omega} \left| \frac{\partial u}{\partial \nu} \right|^2 \langle \nu, V \rangle + \frac{n-2}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{n}{2} |\lambda| \int_{\Omega} |u|^2 dx \\
& - \frac{n-2}{2} \int_{\Omega} |u|^{2^*} dx = \frac{n}{c} \int_{\Omega} \left(\sum_{i,j=1}^n \frac{\partial V_j}{\partial x_i} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \right) dx - \int_{\Omega} |\nabla u|^2 dx.
\end{aligned}$$

Hence

$$\begin{aligned}
& \frac{n}{2c} \int_{\partial\Omega} \left| \frac{\partial u}{\partial \nu} \right|^2 \langle \nu, V \rangle + |\lambda| \int_{\Omega} |u|^2 dx \\
& = \frac{n}{c} \int_{\Omega} \left(\sum_{i,j=1}^n \frac{\partial V_j}{\partial x_i} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \right) dx - \int_{\Omega} |\nabla u|^2 dx \\
& \leq \left(\frac{nM}{c} - 1 \right) \int_{\Omega} |\nabla u|^2 dx \\
& \leq 0.
\end{aligned}$$

Since $\langle \nu, V \rangle > 0$ on $\partial\Omega$, we must have

$$\frac{n}{2c} \int_{\partial\Omega} \left| \frac{\partial u}{\partial \nu} \right|^2 \langle \nu, V \rangle d\sigma + |\lambda| \int_{\Omega} |u|^2 dx = 0.$$

Thus $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, and hence $u \equiv 0$ by the principle of unique continuation. This completes the proof.

References

- [AS] A. Ambroseti and M. Struwe, *A note on the problem $-\Delta u = \lambda u + u|u|^{2^*-2}$* , Manuscripta Math. **54** (1986), 373–379.
- [BN] H. Brezis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical sobolev exponents*, Comm. Pure Appl. Math. **36** (1985), 437–476.
- [L] Z. Liqun, *Uniqueness of positive solutions of $\Delta u + u + u^p = 0$ in a ball*, Comm. Partial Diff. Eq. **17(7&8)** (1992), 1141–1164.

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