Kangweon–Kyungki Math. Jour. 3 (1995), No. 1, pp. 17–20

BASIC PROPERTIES OF BOUNDARY CLUSTER SETS

UN HAING CHOI

Let $w = f(z)$ be a meromorphic function in the unit disc $|z| < 1$. Let $t_0 = e^{i\theta}$ be a fixed point on $\Gamma = \{z : |z| = 1\}$ and A an open arc of Γ containing t_0. We suppose that E is a set of linear measure zero containing t_0 and contained in A. We associate with every $e^{i\theta} \in A - E$ an arbitrary curve Λ_θ in D terminating at $e^{i\theta}$ and the cluster set $C_{\Lambda_\theta}(f, e^{i\theta})$ of $f(z)$ at $e^{i\theta}$ along Λ_θ. Clearly $C_{\Lambda_\theta}(f, e^{i\theta})$ is either a continuum or a single point. We define a new boundary cluster set $C^*_{\Gamma-E}(f, t_0)$ of $f(z)$ at t_0 as follows :

$$C^*_{\Gamma-E}(f, t_0) = \cap_{r>0} M_r$$

where M_r denotes the closure of the union $\cup C_{\Lambda_\theta}(f, e^{i\theta})$ for all $e^{i\theta}$ in the intersection of $A-E$ with $|z-t_0| < r$. As an analogue of this definition we give the following definition. Let $f(z)$ be a meromorphic function in a simply connected domain D, $\widetilde{E}$ a D-coformal null set of prime ends of D such that E the union of impressions of prime ends in $\widetilde{E}$ contains t_0 a boundary point of D. We associate with every accessible boundary point $\mathcal{A}$ with $P(\mathcal{A}) \in \widetilde{D} - \widetilde{E}$ ($\widetilde{D}$ is the set of all prime ends of D) an arc Λ at $P(\mathcal{A})$ in D terminating at $z(\mathcal{A})$ the complex coordinate of $\mathcal{A}$ and the cluster set $C_\Lambda(f, z(\mathcal{A}))$ of $f(z)$ at $z(\mathcal{A})$ along Λ. We define a new boundary cluster set $C^*_{\tilde{D}-\tilde{E},\{\Lambda\}}(f, t_0)$ of f(z) as follows

$$C^*_{\tilde{D}-\tilde{E},\{\Lambda\}}(f, t_0) = \cap_{r>0} M_r$$

where M_r is the closure of the union $\cup C_\Lambda(f, z(\mathcal{A}))$ for all accessible point $\mathcal{A}$ with $P(\mathcal{A}) \in \widetilde{D} - \widetilde{D}$ and $z(\mathcal{A})$ in the disc $|z - t_0| < r$. Clearly we have

$$C^*_{\tilde{D}-\tilde{E},\{\Lambda\}}(f, t_0) \subset C_\Gamma(f, t_0) \subset C_D(f, t_0).$$

We state our main result :

Received July 2, 1994.

THEOREM. *The following three statements are equivalent:*

(A) *Let D be a simply connected domain in the z-plane, which is not the whole plane, and t_0 a boundary point of D, $\widetilde{E}$ a conformal null set of prime ends of D. If $f(z)$ is meromorphic in D and bounded in the intersection of D with some neighborhood of t_0, then*

$$\lim_{z\to t_0}\sup|f(z)| = \limsup_{z(\mathcal{A})\to t_0, P(\mathcal{A})\in\tilde{D}-\tilde{E}} (\inf_{\Lambda}(\limsup_{z\to z(\mathcal{A}), z\in\Lambda} |f(z)|)), \quad (1)$$

where Λ is an arc at an accessible boundary point $\mathcal{A}$ with $P(\mathcal{A})\in \widetilde{D}-\widetilde{E}$ and the convergence is in the ordinary Euclidean metric.

Furthermore, since the left hand side and the right hand denote the radii $r(\widetilde{D}-\widetilde{E},\{\Lambda\})$ of the smallest closed discs with center at $w=0$ which contain $C_D(f,t_0)$ and $C^*_{\tilde{D}-\tilde{E},\{\Lambda\}}(f,t_0)$ defined using, $\{\Lambda : \Lambda \to z(\mathcal{A}), P(\mathcal{A}) \in \widetilde{D}-\widetilde{E}\}$, respectively, the above equality can be written in the form

$$r(D) = r(\widetilde{D}-\widetilde{E},\{\Lambda\})$$

for any choice of $\{\Lambda : \Lambda \to z(\mathcal{A}), P(\mathcal{A}) \in \widetilde{D}-\widetilde{E}\}$. Hence, for fixed $\{\Lambda\}$ we may write

$$r(D) = r(\widetilde{D}-\widetilde{E}) \quad (2)$$

We write $C^*_{\tilde{D}-\tilde{E}}(f,t_0)$ to denote $C^*_{\tilde{D}-\tilde{E},\{\Lambda\}}(f,t_0)$ for fixed Λ.

(B) *If α does not belong to $C_D(f,t_0)$ (in place of the assumption that in (1) that $w=f(z)$ is bounded in the intersection of D with some neighborhood of t_0), then (1) can be replaced by*

$$\rho(C_D(f,t_0),\alpha) = \rho(C^*_{\tilde{D}-\tilde{E}}(f,t_0),\alpha) \quad (3)$$

where $\rho(S,\alpha)$ denotes the spherical distance of α from S.

(C) *Let D be a simply connected domain in the z-plane which is not the whole plane, t_0 a boundary point of D, $\widetilde{E}$ a conformal null set of prime ends of D. If $f(z)$ is a single-valued meromorphic function in D, then*

$$C_D(f,t_0) - C^*_{\tilde{D}-\tilde{E}}(f,t_0)$$

is open, that is,

$$\partial C_D(f,t_0) \subset \partial C^*_{\tilde{D}-\tilde{E}}(f,t_0), \quad (4)$$

where ∂S denotes the boundary of a set S.

Proof. The following proof is a modification of the argument given in Noshiro [1] p. 17.

$(A) \to (B)$: Suppose that α does not belong to $C_D(f, t_0)$ and consider the function $W = F(z)$ obtained by composing a linear transformation

$$W = \frac{(1 + \overline{\alpha} w)}{w - \alpha} \quad with \quad w = f(z).$$

Then (2) holds for $W = F(z)$, that is, the spherical distances of $W = \infty$ from $C_D(F, t_0)$ and $C^*_{\tilde{D}-\tilde{E}}(F, t_0)$ are identical. But since the linear transformation is a rotation of the Riemann sphere, we have (3).

$(B) \to (C)$: Let M and N $(N \subset M)$ be two closed sets in the w-plane. If $\rho(M, w) = \rho(N, w)$ for any point w exterior to M, then we have

$$\partial M \subset \partial N.$$

$(C) \to (A)$: Obviously (4) implies (2).

The proof of (A) was given in [2].

The above result can be used to simplify the proof of the following analogue of Noshiro's theorem [3].

THEOREM. *Let D be a simply connected domain in the z-plane, which is not the whole plane, and let t_0 be a boundary point of D, contained in the union of impressions of prime ends in $\widetilde{E}$, a D-conformal null set. Let $f(z)$ be single-valued and meromorphic in D. If $\alpha \in C_D(f, t_0) C^*_{\tilde{D}-\tilde{E}}(f, t_0)$ is an exceptional value of $f(z)$ in a neighborhood of t_0, then either α is an asymptotic value of $f(z)$ at t_0, or there exists a sequence of points z_n in the boundary of D converging to t_0, such that α is an asymtotic value of $f(z)$ at each z_n.*

References

[1] K. Noshiro, *Cluster sets*, Springer - Verlag, Berlin, 1960.

[2] U. H. Choi, *A Curvilinear extension of Iversen-Tsuji's theorem for a simply connected domain*, Proc. American Math.Soc. **64 No.1** (1977).

[3] U. H. Choi, *On the boundary behavior of analytic functions*, Bulletin of the Korean Math. Soc. **26 No. 2** (1989), 165-168.

Dept. of Mathematics Education
Hong–Ik University
Seoul, 121–791, Korea