▩ 연구논문

Bootstrap Confidence Bounds for $P(X>Y)$

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Abstract

In this paper, the stress strength model is assumed for the populations of X and Y, where distributions of X and Y are independent normal with unknown parameters. We construct bootstrap confidence intervals for reliability, $R=P(X>Y)$ and compare the accuracy of the proposed bootstrap confidence intervals and classical confidence interval through Monte Carlo simulation.

1. Introduction

In many applications, the distributions of the stress and the strength are independent normal with unknown parameters to the investigator. As a specific example, we can consider the rocket-motor experiment data reported by Guttman, Johnson, Bhattacharyya and Reisser (1988). Suppose that one is interested in the reliability of the rocket-motor at the highest operating temperature at which the distribution of operating pressure(Y) tends to be closest to the distribution of chamber burst strength(X).

Church and Harris (1970) and Reisser and Guttman (1986) obtained approximate confidence intervals for R in the stress strength model which X and Y have normal distributions. Guttman, Johnson, Bhattacharyya and Reisser (1988) found an approximate confidence interval for R in stress strength model with explanatory variables. Since the true distribution of the estimator for R is often skewed and biased for a small sample and/or large value of R, the interval based on the asymptotic normal distribution may deteriorate the accuracy. So we will use the bootstrap method to rectify these problems. Efron (1979) initially introduced the bootstrap method to assign the accuracy for an estimator. To a construct approximate confidence interval, Efron (1981, 1982, 1987) and Hall (1988) proposed the percentile method, the bias correct(BC) method, the bias correct acceleration

(BCa) method, and the percentile-t method, etc..

In this paper, we derive large sample property for the bootstrap estimator of R and propose approximate bootstrap confidence intervals for R based on percentile, BC, BCa and percentile-t methods. Also we compare the accuracy of the proposed bootstrap confidence intervals and the approximate confidence interval based on Reisser and Guttman (1986)'s method through Monte Carlo simulation. In particular, we observe the accuracy of these intervals for small sample and/or large value of R.

2. Notations and Preliminaries

In this paper the following notations are used.

$N(a, b)$: normal distribution with mean a and variance b.

X, Y : strength and stress of component having $N(\mu_x, \sigma_x^2)$ and $N(\mu_y, \sigma_y^2)$, respectively.

$F(x), G(y)$: distribution functions(df's) of strength and stress, respectively.

$\boldsymbol{X} = (X_1, X_2, \cdots, X_{n_1})$: random samples from $F(x)$.

$\boldsymbol{Y} = (Y_1, Y_2, \cdots, Y_{n_2})$: random samples from $G(y)$.

$\boldsymbol{X}^* = (X_1^*, X_2^*, \cdots, X_{n_1}^*)$: bootstrap samples from sample df's of X.

$\boldsymbol{Y}^* = (Y_1^*, Y_2^*, \cdots, Y_{n_2}^*)$: bootstrap samples from sample df's of Y.

$\overline{X}, \overline{Y}$: sample means for X and Y, respectively.

$\hat{\sigma}_x^2, \hat{\sigma}_y^2$: sample variances for X and Y, respectively.

$\Phi(\cdot)$: cdf of the standard normal distribution.

$z^{(\alpha)}$: $\Phi(z^{(\alpha)}) \equiv \alpha$

$\hat{\theta}^*$: bootstrap version of $\hat{\theta}$ for any bootstrap replication.

Under the stress strength model with independent normal distributions, it is known that the reliability becomes $R = P(X>Y) = \Phi(\rho)$, where $\rho = (\mu_x - \mu_y)/\sqrt{\sigma_x^2 + \sigma_y^2}$. Reisser and Guttman (1986) obtained the estimator $\hat{R}$ of R given as

$$\hat{R} = \Phi(\hat{\rho}) = \Phi((\overline{X}-\overline{Y})/\sqrt{\hat{\sigma}_x^2 + \hat{\sigma}_y^2}) \tag{2.1}$$

where $\hat{\rho} = (\overline{X}-\overline{Y})/\sqrt{\hat{\sigma}_x^2 + \hat{\sigma}_y^2}$. Also they proved that $\hat{\rho}$ has the asymptotic normal distribution with mean ρ and variance $\sigma_\rho^2=1/M+\rho^2/2f$, where $M=(\sigma_x^2+\sigma_y^2)/(\sigma_x^2/n_1+\sigma_y^2/n_2)$ and $f=(\sigma_x^2+\sigma_y^2)^2/(\sigma_x^4/(n_1-1)+\sigma_y^4/(n_2-1))$. The asymptotic variance of $\hat{\rho}$ is estimated by using $\hat{\sigma}_x^2$ and $\hat{\sigma}_y^2$ instead of σ_x^2 and σ_y^2, respectively, in the formulas for M and f.

Hence they show that a $100(1-2\alpha)\%$ approximate confidence interval for R is given by

$$(\Phi(\hat{\rho}+z^{(\alpha)}\cdot\hat{\sigma}_\rho),\ \Phi(\hat{\rho}+z^{(1-\alpha)}\cdot\hat{\sigma}_\rho)). \tag{2.2}$$

3. Consistency for Bootstrap Estimator

The bootstrap procedure is a resampling scheme that one attempts to learn the sampling properties of a statistic by recomputing its value on the basis of a new sample realized from the original one. The bootstrap procedure for construction of bootstrap estimator for R provides confidence interval estimates by using the plug-in principle as follows:

(1) Compute the plug-in estimates of μ_x, μ_y, σ_x^2 and σ_y^2 given by $\overline{X}$, $\overline{Y}$, $S_x^2=n_1^{-1}\sum_{i=1}^{n_1}(X_i-\overline{X})^2$ and $S_y^2=n_2^{-1}\sum_{j=1}^{n_2}(Y_j-\overline{Y})^2$ from $\boldsymbol{X}$ and $\boldsymbol{Y}$, respectively.

(2) Construct the sampling distribution $\hat{F}_{n_1}^*$ and $\hat{G}_{n_2}^*$ (from $\boldsymbol{X}$ and $\boldsymbol{Y}$) based on $\overline{X}$, $\overline{Y}$, S_x^2 and S_y^2, respectively. That is, $\hat{F}_{n_1}^*\sim N(\overline{X}, S_x^2)$ and $\hat{G}_{n_2}^*\sim N(\overline{Y}, S_x^2)$.

(3) Generate B random samples of size n_1 and n_2 from fixed $\hat{F}_{n_1}^*$ and $\hat{G}_{n_2}^*$, respectively. We call the corresponding samples by the *bootstrap samples*, and denote $\boldsymbol{X}^{*b}=(X_1^{*b}, X_2^{*b}, \cdots, X_{n_1}^{*b})$ and $\boldsymbol{Y}^{*b}=(Y_1^{*b}, Y_2^{*b}, \cdots, Y_{n_2}^{*b})$, where $b=1, 2, \cdots, B$.

(4) Compute $\hat{R}^{*b}=\Phi(\hat{\rho}^{*b})$, where $\hat{\rho}^{*b}=(\overline{X}^{*b}-\overline{Y}^{*b})/\sqrt{S_x^{2*b}+S_y^{2*b}}$. We call $\overline{X}^{*b}$, $\overline{Y}^{*b}$, S_x^{2*b} S_y^{2*b} and $\hat{R}^{*b}$ by *bootstrap estimators* for μ_x, μ_y, σ_x^2, σ_y^2 and R, respectively.

Theorem. For given X and Y, suppose that $\boldsymbol{X}^*$ and $\boldsymbol{Y}^*$ are the bootstrap samples of sizes n_1 and n_2 from the sample distribution functions $\hat{F}_{n_1}^*$ and $\hat{G}_{n_2}^*$, respectively. Then the bootstrap estimator $\hat{R}^*$ is a consistent estimator of R.

Proof. For an arbitrary positive ε,

$$\begin{aligned} P(|\overline{X}^* - \overline{X}| \geq \varepsilon/2) &\leq E(\overline{X}^* - \overline{X})^2/(\varepsilon/2)^2 \\ &= E[E[(\overline{X}^* - \overline{X})^2 \mid \underline{X}]]/(\varepsilon/2)^2 \\ &= E(S_x^2)/(n_1(\varepsilon/2)^2) \\ &= (n_1 - 1)\sigma^2/(n_1^2(\varepsilon/2)^2) \rightarrow 0, \quad \text{as} \quad n_1 \rightarrow \infty. \end{aligned}$$

Also,

$$\begin{aligned} P(|S_x^{2*} - S_x^2| \geq \varepsilon/2) &\leq E(S_x^{2*} - S_x^2)^2/(\varepsilon/2)^2 \\ &= E[E[(S^{2*} - S^2)^2 \mid \underline{X}]]/(\varepsilon/2)^2 \\ &= 2(n_1 - 1)E(S_x^4)/(n_1^2(\varepsilon/2)^2) \\ &= 8(n_1 - 1)(n_1^2 - 1)\sigma_x^4/(n_1^4\varepsilon^2) \rightarrow 0, \quad \text{as} \quad n_1 \rightarrow \infty. \end{aligned}$$

Since it is known that $\overline{X}$ and S_x^2 are a consistent estimators for μ_x and σ_x^2, respectively, $\overline{X}^*$ and S_x^{2*} are a consistent estimators for μ_x and σ_x^2, respectively. Similarly, $\overline{Y}^*$ and S_y^{2*} are consitent estimators for μ_y and σ_y^2, respectively. Hence $\hat{\rho}^* = (\overline{X}^* - \overline{Y}^*)/\sqrt{S_x^{2*} + S_y^{2*}}$ is a consistent estimator of ρ. Since Φ is continuous function, $\hat{R}^*$ is a consistent estimator of R.

Note that, under the assumptions of theorem, the asymptotic distribution of $\hat{R}^*$ and $\hat{R}$ are same.

4. Bootstrap Confidence Intervals for Reliability

In this section we construct approximate bootstrap confidence intervals for R. All confidence intervals are two-sided and equal-tailed with confidence level 100 $(1-2\alpha)\%$.

4.1 Percentile method

The confidence interval by the bootstrap percentile method(percentile interval is obtained by percentiles of the empirical bootstrap distribution of $\hat{R}^*$. Let $\hat{H}^*$ be the empirical cumulative distribution function of $\hat{R}^*$. Then it is constructed by $\hat{H}^*(s) = B^{-1}\sum_{b=1}^{B} I(\hat{R}^{*b} \leq s)$, where s is an arbitrary real value and $I(\cdot)$ is an indicator function. Let $\hat{H}^{*-1}(\alpha)$ be a 100α empirical percentile of $\hat{R}^*$ given by

$$\hat{H}^{*-1}(\alpha) = \inf\{s : \hat{H}^*(s) \geq \alpha\}. \qquad (4.1)$$

That is, $\hat{H}^{*-1}(\alpha)$ is the $B\alpha$th value in the ordered list of the B replications of $\hat{R}^{*b}$. If $B\alpha$ is not an integer, we can take the largest integer that less than or equal to $(B+1)\alpha$. Then a $100(1-2\alpha)\%$ percentile interval for R is approximated by

$$(\hat{H}^{*-1}(\alpha),\ \hat{H}^{*-1}(1-\alpha)). \tag{4.2}$$

4.2 Bias correct method

The BC method adjusts a possible bias in estimating R. The bias correction is given by

$$\hat{z}_0 = \Phi^{-1}(\hat{H}^*(\hat{R})) = \Phi^{-1}\left[B^{-1}\sum_{b=1}^{B} I(\hat{R}^{*b} \le \hat{R})\right], \tag{4.3}$$

where $\Phi^{-1}(\cdot)$ indicates the inverse function of the standard normal cumulative distribution function. That is, $\hat{z}_0$ is the discrepancy between the medians of $\hat{R}^*$ and $\hat{R}$ in normal unit. Therefore, we have a $100(1-2\alpha)\%$ approximate BC interval for R given by

$$(\hat{H}^{*-1}(\alpha_1),\ \hat{H}^{*-1}(\alpha_2)), \tag{4.4}$$

where $\alpha_1 = \Phi(2\hat{z}_0 + z^{(\alpha)})$ and $\alpha_2 = \Phi(2\hat{z}_0 + z^{(1-\alpha)})$.

4.3 Bias correct acceleration method

The BCa method corrects both the bias and standard error for $\hat{R}$. The confidence interval by BCa method(BCa interval) requires to calculate the bias-correction constant $\hat{z}_0$ and the acceleration constant $\hat{a}$. In fact, the bias-correction constant $\hat{z}_0$ is the same as that of BC method. And $\hat{a}$, measured on a normalized scale, refers to the rate of change of the standard error of $\hat{R}$ with respect to the true reliability R.

For the parametric bootstrap method, all calculations relate only to the sufficient statistics $\overline{X}$, S_x^2, $\overline{Y}$ and S_y^2 for μ_x, σ_x^2, μ_y and σ_y^2, respectively. Of course, $\overline{X}$, S_x^2, $\overline{Y}$ and S_y^2 are distributed as $N(\mu_x, \sigma_x^2/n_1)$, $(\sigma_x^2/n_1)\chi^2(n_1-1)$, $N(\mu_y, \sigma_y^2/n_2)$ and $(\sigma_y^2/n_2)\chi^2(n_1-1)$, respectively. Also, $\overline{X}$, S_x^2, $\overline{Y}$ and S_y^2 are stochastically independent. Let $\hat{\underline{\eta}}' = (\overline{X}, S_x^2, \overline{Y}, S_y^2)$ and $\underline{\eta}' = (\mu_x, \sigma_x^2, \mu_y, \sigma_y^2)$. Then the joint probability density function of $\hat{\underline{\eta}}'$ can be written as

$$f_{\underline{\eta}}(\hat{\underline{\eta}}') = f_0(\hat{\underline{\eta}}')\exp[g_0(\hat{\underline{\eta}}', \underline{\eta}') - \Psi_0(\underline{\eta}')], \tag{4.5}$$

where $f_0(\hat{\underline{\eta}}') = [\,2\pi\Gamma((n_1-1)/2)\cdot\Gamma((n_2-1)/2)2^{(n_1+n_2-2)/2}\,]^{-1}\cdot n_1^{n_1/2}\,n_2^{n_2/2}$,

$$g_0(\underline{\eta}', \hat{\underline{\eta}}') = (2n_1\mu_x\overline{X} - n_1\overline{X}^2 - n_1S_x^2)/2\sigma_x^2 + (2n_2\mu_y\overline{Y} - n_2\overline{Y}^2 - n_2S_y^2)/2\sigma_y^2$$
$$+ (n_1-3)/2\cdot\log(S_x^2) + (n_2-3)/2\cdot\log(S_y^2),$$

and

$$\Psi_0(\underline{\eta}') = n_1\mu_x^2/2\sigma_x^2 + n_1\log(\sigma_x^2)/2 + n_2\mu_y^2/2\sigma_y^2 + n_2\log(\sigma_y^2)/2.$$

For multiparameter family case, we will find $\hat{a}$ following Stein's construction (1956). That is, we replace the multiparameter family $\mathfrak{I} = \{f_{\underline{\eta}}(Z)\}$ by the least favorable one parameter family $\hat{\mathfrak{I}} = \{\hat{f}_{\underline{\eta}}(Z) \equiv f_{\hat{\underline{\eta}}+\lambda\hat{\underline{w}}}(Z)\}$, where $Z=(X, Y)$. Then we first obtain $\hat{\underline{w}}$ such that the least favorable direction at $\underline{\eta}=\hat{\underline{\eta}}$ is defined to be $\hat{\underline{w}} = (\mathcal{L}^{-}_{\hat{\underline{\eta}}})^{-1}\hat{\nabla}_{\hat{\underline{\eta}}}$, where $\mathcal{L}^{-}_{\hat{\underline{\eta}}}$ is Fisher information matrix and $\hat{\nabla}_{\hat{\underline{\eta}}}$ is the gradient of ρ given by $\hat{\nabla}_{\hat{\underline{\eta}}} = \frac{\partial\rho}{\partial\underline{\eta}}\big|_{\underline{\eta}=\hat{\underline{\eta}}}$. After some algebraic calculation, we have

$$\mathcal{L}^{-}_{\hat{\underline{\eta}}} = \begin{bmatrix} n_1/S_x^2 & 0 & 0 & 0 \\ 0 & n_2/S_y^2 & 0 & 0 \\ 0 & 0 & n_1/(2S_x^2) & 0 \\ 0 & 0 & 0 & n_2/(2S_y^2) \end{bmatrix},$$

and

$$\hat{\nabla}_{\hat{\underline{\eta}}} = \begin{bmatrix} 1/\sqrt{S_x^2+S_y^2} \\ -1/\sqrt{S_x^2+S_y^2} \\ -(\overline{X}-\overline{Y})/2(S_x^2+S_y^2)^{3/2} \\ -(\overline{X}-\overline{Y})/2(S_x^2+S_y^2)^{3/2} \end{bmatrix}$$

Hence, we have $\hat{\underline{w}}' = (W_1, W_2, W_3, W_4)$, where $W_1 = (S_x^2/(n_1\sqrt{S_x^2+S_y^2})$, $W_2 = -S_y^2/(n_2\sqrt{S_x^2+S_y^2})$, $W_3 = -S_x^4(\overline{X}-\overline{Y})/(n_1(S_x^2+(S_y^2)^{3/2})$, and $W_4 = -S_y^4(\overline{X}-\overline{Y})/(n_2(S_x^2+(S_y^2)^{3/2})$. By the method of Efron (1987), $\hat{a}$ can be obtained as the following

$$\hat{a} = \frac{1}{6}\cdot\frac{\hat{\Psi}^{(3)}(0)}{(\hat{\Psi}^{(2)}(0))^{(3/2)}}, \tag{4.6}$$

where $\hat{\Psi}^{(j)}(0) = \dfrac{\partial^j \Psi_0(\hat{\eta}+\lambda\hat{w})}{\partial\lambda^j}\Big|_{\lambda=0}$. Calculating $\hat{\Psi}^{(j)}(\cdot)$ and $\hat{w}$, we can obtain

$$\hat{\Psi}^{(2)}(0) = 2^{-1} n_1[-W_3^2/S_x^2 + \{2(W_1 S_x^2)^2 - 4W_1 W_3 \overline{X} S_x^2 + 2(W_3\overline{X})^2\}/S_x^6] + 2^{-1} n_2[-W_4^2/S_y^4 + \{2(W_2 S_y^2)^2 - 4W_2 W_4 \overline{Y} S_y^2 + 2(W_4\overline{Y})^2\}/S_y^6]$$

and

$$\hat{\Psi}^{(3)}(0) = 2^{-1} n_1(-2W_3^3/S_x^6 + A) + 2^{-1} n_2(-2W_4^3/S_y^6 + B).$$

where $A = \{-6(W_1 S_x^2)^2 W_3 + 12W_1 W_3^2 \overline{X} S_x^2 - 6W_3^3 \overline{X}^2\}/S_x^8$
and

$$B = \{-6(W_2 S_y^2)^2 W_4 + 12W_2 W_4^2 \overline{Y} S_y^2 - 6W_4^3 \overline{Y}^2\}/S_y^8.$$

Therefore, we have a $100(1-2\alpha)\%$ approximate BCa interval for R by

$$(\Phi(\hat{H}^{*-1}(\alpha_3)),\ \Phi(\hat{H}^{*-1}(\alpha_4))), \tag{4.7}$$

where $\alpha_3 = \Phi[\hat{z}_0 + (\hat{z}_0 + z^{(\alpha)})/(1-\hat{a}(\hat{z}_0 + z^{(\alpha)}))]$

and

$$\alpha_4 = \Phi[\hat{z}_0 + (\hat{z}_0 + z^{(1-\alpha)})/(1-\hat{a}(\hat{z}_0 + z^{(1-\alpha)}))].$$

4.4 Percentile-t method

The confidence interval by the bootstrap percentile-t method (percentile-t interval) is constructed by using the bootstrap distribution of an approximately pivotal quantity for $\hat{\rho}$ instead of $\hat{H}^*$. We define an approximate bootstrap pivotal quantity for $\hat{\rho}$ by

$$\hat{\rho}^*_{\mathrm{STUD}} = (\hat{\rho}^* - \hat{\rho})/\hat{\sigma}^*_{\rho}, \tag{4.8}$$

where $\hat{\rho}^*$ and $\hat{\sigma}^*_{\rho}$ are the bootstrap versions of $\hat{\rho}$ and $\hat{\sigma}_{\rho}$, respectively. We compute the empirical distribution function $\hat{H}^*_{\mathrm{STUD}}$ of $\hat{\rho}^*_{\mathrm{STUD}}$ by

$$\hat{H}^*_{\mathrm{STUD}}(s) = B^{-1} \sum_{b=1}^{B} I(\hat{\rho}^{*b}_{\mathrm{STUD}} \le s), \tag{4.9}$$

for all s. Let $\hat{H}^{*-1}_{STUD}$ denote 100α empirical percentile of $\hat{\rho}^*_{STUD}$, computed by

$$\hat{H}^{*-1}_{STUD}(\alpha) = \inf\{s : \hat{H}^{*-1}_{STUD}(s) \geq (\alpha)\}. \tag{4.10}$$

Here, $\hat{H}^{*-1}_{STUD}(\alpha)$ is the $B\alpha$th value in the ordered list of the B replications of $\hat{\rho}^*_{STUD}$. Then we have a $100(1-2\alpha)\%$ approximate percentile-t interval for R by

$$(\Phi(\hat{\rho} + \hat{\sigma}_\rho \cdot \hat{H}^{*-1}_{STUD}(\alpha)),\ \Phi(\hat{\rho} + \hat{\sigma}_\rho \cdot \hat{H}^{*-1}_{STUD}(1-\alpha))). \tag{4.11}$$

5. Comparisons and Conclusions

To compare the approximate bootstrap confidence interval estimates with the confidence interval estimate based on asymptotic normal distribution, the coverage probabilities and the interval length for these intervals are computed by Monte Carlo simulation. We consider that the true reliabilities of R are 0.3, 0.5, 0.7 and 0.9, sample sizes n_1+n_2 are 5, 10, 20 and 50, and the confidence level $(1-2\alpha)$ is 0.90. For given independent random samples the approximate confidence intervals are constructed by each method with bootstrap replications $B=1000$ times.

Through Tables 1 and 2, one can observe the following facts :

(1) From the viewpoint of coverage probabilities, the bias correction, bias correct acceleration, and percentile-t intervals work better than Reisser and Guttman's (RG) interval, but the percentile interval is not.

(2) The interval length for all approximate confidence intervals tend to decrease as R deviates from 0.5. As a whole, the value of RG interval length become shorter than those of the bootstrap interval lengths.

(3) The coverage probabilities for all approximate intervals converge to true confidence level, $(1-2\alpha)$. And the differences of all interval lengths tend to be reduced.

〈 Table 1 〉 Coverage Probabilities for Confidence Level 0.90

SAMPLE SIZE	R	RG	Percentile	BC	BCa	Percentile-t
5	0.3	.8520	.7300	.9140	.9000	.9200
	0.5	.8500	.7600	.9060	.8920	.8840
	0.7	.8600	.7160	.9200	.8900	.8900
	0.9	.8500	.6200	.8600	.8560	.9260
10	0.3	.8680	.8220	.9180	.9020	.9220
	0.5	.8740	.8360	.9380	.9140	.9080
	0.7	.8540	.7900	.9060	.8900	.9000
	0.9	.8700	.7420	.9060	.8960	.9240
20	0.3	.8600	.8300	.8860	.8720	.9040
	0.5	.8720	.8560	.8980	.8860	.8980
	0.7	.8820	.8340	.9060	.8920	.9160
	0.9	.8780	.7760	.8880	.8840	.9120
50	0.3	.8800	.8840	.9100	.9080	.8960
	0.5	.8820	.8720	.9040	.9000	.9040
	0.7	.8880	.8620	.9000	.9000	.9040
	0.9	.8840	.8740	.8920	.8940	.9020

〈 Table 2 〉 Interval Lengths for Confidence Level 0.90

SAMPLE SIZE	R	RG	Percentile	BC	BCa	Percentile-t
5	0.3	.5463	.5527	.7062	.6691	.5845
	0.5	.5844	.6611	.7757	.7491	.6239
	0.7	.5437	.5535	.7189	.6866	.5803
	0.9	.3794	.2402	.4451	.4057	.4161
10	0.3	.4577	.4859	.5372	.5254	.4991
	0.5	.4996	.5690	.6026	.5950	.5424
	0.7	.4523	.4781	.5363	.5245	.4937
	0.9	.2989	.2406	.3337	.3177	.3380
20	0.3	.3439	.3590	.3732	.3696	.3647
	0.5	.3821	.4165	.4216	.4201	.4039
	0.7	.3420	.3572	.3730	.3695	.3621
	0.9	.2028	.1790	.2128	.2079	.2206
50	0.3	.2555	.2623	.2667	.2658	.2634
	0.5	.2815	.2943	.2960	.2959	.2896
	0.7	.2519	.2583	.2631	.2618	.2595
	0.9	.1530	.1447	.1568	.1549	.1602

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