▩ 연구논문

Accelerated Life Test Plans Based on Small Sample Property +

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Abstract

This paper suggests optimal accelerated constant stress life tests in Exponential distribution. The relationship between the log-mean life and the loaded stress is assumed to be linear.

Optimal plans considering mean square errors of maximum likelihood estimators of the log mean life and test costs are obtained. We consider accelerated life tests with two stress levels, and as data types, failure censoring(type II) and time censoring(type I) data are used. We propose the procedure to obtain the optimal plans for each case. Some examples are also included.

1. Introduction

Accelerated life testing provides quick information on life distribution of products. The more reliable a product is, the more difficult it is to estimate its reliability because we can not sometimes obtain the failures for a long time in life testing. Even if such failures are obtained, the information would be obsolete because of short life cycle. In this situation, one approach to overcome the difficulty is to use accelerated life test, in which units are tested at higher stress levels than normal operating condition and from the failure data under accelerated stress we can predict the performance of the product at the normal stress level. There are various statistical and engineering problems in accelerated life tests. For example, statistical inferences about model parameters of relationship between life time distribution and stress-life distribution, or test stress levels. This paper considers the design problem in constant-stress accelerated life tests

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Therefore we limit our explanation and literature survey to constant-stress accelerated life test. For other stress loading methods, you can refer a good book, Nelson(1990). Traditional plans in constant-stress accelerates life tests were equally spaced test stresses and the same number of test units allocated to each test stress. The plans have been shown to be inefficient. Therefore, a lot of authors consider other plans to minimize or maximize the specified optimality criteria. The usual optimality criterion is the asymptotic variance of maximum likelihood estimator of the mean or specific percentile of life distribution at the design stress. Chernoff(1962) studied optimal accelerated life test plan for exponential distribution in which the failure rate is a quadratic function of the stresses. Yum & Choi(1989) considered the optimal test plans under periodic inspection. un & Pan(1994) studied an economic design of accelerated life test in exponential case. You can refer a good survey book (Nelson(1990)) for constant-stress accelerated life test designs for other distributions (Weibull, log-normal distribution).

This paper consider optimum plans for estimating the mean life of an exponential distribution at the design stress where the optimality criterion is mean square error of maximum likelihood estimation of mean at design stress. We obtain the mean square errors and propose the procedures for obtaining the optimal designs. Some examples are also studied.

2. The Model

Notation

L log-likelihood

n total number of test units

n_i number of units allocated to each stress $i=1, 2$

π_i n_i/n

x_i test stress $i=0, 1, 2$

x_0, x_1, x_2 design stress, lower stress, higher stress

η censoring time (For type I censoring)

r censoring number (For type II censoring)

r_i number of units failed at each stress level

t_{ij} jth failure time at i stress level $i=1, 2,\ j=1, \cdots, n_i$

T_i total tume on n test at i stress level

$T_i = \sum t_{ij} + (n_i - r_i)\eta$, where type I censored data

$T_i = \sum t_{ij} + (n_i - r_i)t_{ir}$, where type II censored data,

γ_i	life-stress relationship parameters $i=0, 1$
β_i	standardized parameters of life-stress relationship $i=0, 1$
μ	mean life
ξ_i	standardized stress $i=0, 1, 2$
C_a	slope of function of test cost per unit time
C_b	intersection of function of test cost per unit time

Test Procedure

1. n_1, units chosen randomly from n units are allocated to the low test stress x_1 and the remaining n_2 units to the high test stress x_2.
2. Each unit is tested until censoring time η if it does not fail (type I censoring) or each unit is tested until the number of failed unit is r (type II censoring)

Assumptions

1. The high test stress x_2 is specified (given constant).
2. The time to the failure of the any unit is independent to the failure time of the other unit
3. For any stress, the life distribution is exponential.
4. The log mean life $\ln \mu(x)$ is a linear function of the stress,

$$\ln \mu(x) = \gamma_0 + \gamma_1 x$$

   where γ_0, γ_1 are unknown parameters and should be estimated.

5. Test cost per unit time is a linear function of stress,

$$C_i = C_b - C_a \xi_i$$

   and test cost per unit time is incresing linearly in stress level.

6. The censoring time and number of units failed are given.

Estimation Method-Maximum Likelihood Estimation

At first, to simplify the estimation problem, we standardize the stress level as follows:

$$x = x_2 - \xi(x_2 - x_1)$$

and

$$\xi(x) = (x_2 - x)/(x_2 - x_0)$$

so that $\xi(x_0)=1$, $\xi(x_2)=0$, and let $\xi(x_1)=\xi_1$.
Thus $\ln\mu(\xi)=\beta_0+\beta_1\xi$, where $\beta_0=\gamma_0+\gamma_1 x_2$, $\beta_1=-\gamma_1(x_2-x_0)$
Our objective is to estimate $\ln\mu(\xi=1)$, the log mean life at x_0, and the maximum likelihood estimation is used as estimation method.
The log-likelihood is for two censoring types,

$$L = -r_2\beta_0 - T_2/\exp(\beta_0) - r_1(\beta_0+\beta_1\xi_1) - T_1/\exp(\beta_0+\beta_1\xi_1)$$

Where T_i is total time on tests and its form is different based in censoring type. By differentiating L with respect to β_0 and β_1 there maximum likelihood estimators are as follows;

$$\hat{\beta}_0 = \ln(T_2/r_2)$$
$$\hat{\beta}_1 = 1/[\xi_1\{\ln(T_1/r_1) - \ln(T_2/r_2)\}]$$

Therefore, the maximum likelihood estimator of log mean life at design stress is,

$$\ln\hat{\mu}(1) = \hat{\beta}_0+\hat{\beta}_1 = \ln(T_2/r_2)+1/[\xi_1\{\ln(T_1/r_1)-\ln(T_2/r_2)\}]$$

Optimum Plan

The objective of the accelerated life test is to estimate $\ln\mu(1)$. Mean square error of $\ln\mu(1)$ may be the inverse of amount of information. Thus the problem minimizing the multiplication of the mean square error and total test cost is equal to the problem maximizing the amount of information per unit cost. We are going to obtain the test plan to minimize the multiplication of themean squre error and total test costs.

So, the proposed optimization criterion consider mean square error of $\hat{\beta}_0+\hat{\beta}_1$ and test costs.

Therefore the optimization problem can be summarized as follows:

$$\min_{\xi_1,\pi_1}(E[\hat{\beta}_0+\hat{\beta}_1-\beta_0-\beta_1]^2 \times (C_1E[T_1]+C_2E[T_2]) \qquad (1)$$
$$0 \le \xi_1 \le 1$$
$$0 \le \pi_1 \le 1$$

The previous objective function changes as the type of data. So we will specify the special objective functions for the two censoring data.

Failure censoring data case (Type II Censoring)

In failure censoring data case, $2T_i/\theta_i$ follows chi-square distribution with degree of freedom, 2r. From the previous fact, we can obtain $E[\ln(\frac{T_i}{r_i})]$, $E[\ln(\frac{T_i}{r_i})]^2$ numerically,

$$E[\ln(\frac{T_i}{r_i})] = \int_0^\infty \ln t \times \frac{t^{r/2-1}}{\Gamma(r/2)2^{r/2}} \exp[-t/2]dt - \ln[2/\theta_i] - \ln r_i$$

$$E[\ln(\frac{T_i}{r_i})]^2 = \int_0^\infty \ln^2 t \times \frac{t^{r/2-1}}{\Gamma(r/2)2^{r/2}} \exp[-t/2]dt - 2\{\ln[\frac{2}{\theta_i}] + \ln r_i\}\{E[\ln \frac{T_i}{r_i}] + \ln r_i\} + (\ln^2[2/\theta_i] + \ln^2 r_i)$$

The expected total test time at i stress level is

$$E[T_i] = \theta_i \sum_{i=1}^{r} \sum_{j=1}^{k} \frac{1}{n_i - j + 1} + (n_i - r)\theta_i \sum_{j=1}^{r} \frac{1}{n_i - j + 1}$$

where $\theta_1 = \exp(\beta_0 + \beta_1 \xi)$ and $\theta_2 = \exp(\beta_0)$.

The optimization problem is

$$\underset{\xi_1 \cdot n_1}{\mathrm{Min}}(E[\ln \frac{T_2}{r_2} + \frac{1}{\xi_1}\{\ln \frac{T_1}{r_1} - \ln \frac{T_2}{r_2}\} - \beta_1 - \beta_0]^2 \times (C_1 E[T_1] + C_2 E[T_2])) \quad (2)$$

It is difficult to derive the object function as a closed form and to prove the convexity of object function analytically, and we should use the search method for optimization of optimal design.

⟨Example 1⟩ The values of model parameters and cost parameters are given as follows (Miller and Nelson(1983)).

$\beta_0 = 4.6972$ $\quad \beta_1 = 7.1784$
$C_a = 18$(Won/unit time) $\quad C_b = 36$(Won/unit time)
$n = 20$ $\quad r = 3$.

For given n_i (3~17), we investigated the shapes of the objective functions and obtained optimal ξ^*. The objective function was a convex function for given n_i.

Finally the optimal solution is given by

$\xi_1^* = 0.37$
$n_1^* = 15$
total test cost = 153100(Won)
mean square error = 14.17
and the value of objective function = 14.17 × 15310 = 2169681

Therefore we must assign 15 units among total 20 units to low stress level of which the stress level is $0.37x_0 + 0.63x_2$. Which is a linear combination of design and high stress level.
β_1 is more importance parameter and we try to derive the effect of incorrect estimation of β_1. and from the below table we can find the robustness of test plan to β_1 preestimator.

β_1	5.1	6.1	7.1784	8.1	9.1
ratio of object function to correct estimation	1.24	1.04	1.0	1.03	1.09

Time censoring data case (Type I censoring)
In time censoring data case, Bartholomew(1963) showed that T_i/r_i follows distribution

$$p(t) = \frac{1}{1-e^{-n_i\eta/\theta}} \sum_{k=1}^{n_i} \binom{n_i}{k} \frac{(k/\theta)^k}{\Gamma(k)} e^{-kt/\theta} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \langle t - \frac{\eta}{k}(n_i - k + j) \rangle^{k-1}$$

The symbol $\langle - \rangle$ means that the expression is to be taken as zero if the contents are negative.
where $\theta_1 = \exp(\beta_0 + \beta_1\xi)$ and $\theta_2 = \exp(\beta_0)$.

From the existing result, we can obtain $E[\ln(\frac{T_i}{r_i})]$, $E[\ln(\frac{T_i}{r_i})]^2$ numerically,

$$E[\ln\frac{T_i}{r_i}] = \int_0^\infty \ln(t) \times \frac{1}{1-e^{-n_i\eta/\theta}} \sum_{k=1}^{n_i} \binom{n_i}{k} \frac{(k/\theta)^k}{\Gamma(k)} e^{-kt/\theta} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \langle t - \frac{\eta}{k}(n_i - k + j) \rangle^{k-1} dt$$

$$E[\ln\frac{T_i}{r_i}]^2 = \int_0^{\eta} \ln^2(t) \times \frac{1}{1-e^{-n_i\eta/\theta}} \sum_{k=1}^{n_i} \binom{n_i}{k} \frac{(k/\theta)^k}{\Gamma(k)} e^{-kt/\theta} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \langle t - \frac{\eta}{k}(n_i - k + j) \rangle^{k-1} dt$$

and the expected total test time at i stress level is

$$E[T_i] = \{ \frac{-\eta \exp(-\eta/\theta_i) + \theta_i (1 - \exp(-\eta/\theta_i))}{1 - \exp(-\eta/\theta_i)} \}$$

the object function is,

$$\min_{\xi_1, n_1} (E[\ln \frac{T_2}{r_2} + \frac{1}{\xi_1}\{ \ln \frac{T_1}{r_1} - \ln \frac{T_2}{r_2} \} - \beta_1 - \beta_0]^2 \times (C_1 E[T_1] + C_2 E[T_2])) \qquad (3)$$

can be obtained numerically as same as the type II censoring case.

⟨Example 2⟩ The values of model parameters and cost parameters are given as follows.

β_0 = 4.6972 β_1 = 7.1784
C_a = 18(Won/unit time) C_b = 36(Won/unit time)
n = 20 η = 1000(unit time).

The optimal solution is given by
ξ_1^* = 0.59
n_1^* = 9
total test cost = 155071(Won)
mean square error = 2.076
and value of objective function = 2.076 × 155071 = 321927

Therefore we assign 3 units among total 20 units to low stress level test of which the stress level is $0.59x_0 + 0.41x_2$. Which is a linear combination of design and high stress level.
β_1 is more importance parameter and we try to derive the effect of incorrect estimation of β_1. and from the below table we can find the objective function sensitive to β_1 preestimator. Thus, for optimal design, we should know the value of β_1 very exactly.

β_1	5.1	6.1	7.1784	8.1	9.1
ratio of object function to correct estimation	1.8	1.5	1.0	1.5	1.7

3. Conclusion

This paper considered optimum plans for estimating the mean life of an exponential distribution at the design stress where the optimality criterion consider mean square error of maximum likelihood estimation of mean at design stress and test cost. We obtain the mean square error and propose the procedure for obtaining the optimal designs. We consider two data cases; failure censoring data case, time censoring case (The complete test case is summarized in Appendix). We can find that it is difficult to obtain the objective functions and to prove the convexity analytically. So we obtained the optimal solutions numerically. We showed examples at each case.

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Appendix (complete case)

In complete case, tests are continued until all unit are failed. $2T_i/\theta_i$ follows gamma distribution with (n_i, θ_i). From the previous fact, we can obtain $E[\ln(\frac{T_i}{r_i})]$, $E[\ln(\frac{T_i}{r_i})]^2$ numerically,

$$E[\ln(\frac{T_i}{r_i})] = \int_0^\infty \ln t \times \frac{t^{n_i-1}}{\Gamma(n_i)\theta_i^k} \exp[-t/\theta_i]dt$$

$$E[\ln(\frac{T_i}{r_i})]^2 = \int_0^\infty \ln^2 t \times \frac{t^{n_i-1}}{\Gamma(n_i)\theta_i^k} \exp[-t/\theta_i]dt$$

The expected total test time at i stress level is

$$E[T_i] = \theta_i \sum_{k=1}^{n_i} \sum_{j=1}^{k} \frac{1}{n_i - j + 1}$$

where $\theta_1 = \exp(\beta_0 + \beta_1 \xi)$ and $\theta_2 = \exp(\beta_0)$.

Finally, the optimization problem is given by

$$\underset{\xi_1, \pi_1}{\text{Min}} \left(E[\ln \frac{T_2}{r_2} + \frac{1}{\xi_1}\{\ln \frac{T_1}{r_1} - \ln \frac{T_2}{r_2}\} - \beta_1 - \beta_0]^2 \times (C_1 E[T_1] + C_2 E[T_2])\right)$$

The objective function can be obtained numerically easier than the censoring cases.