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On Fuzzy Quotient Semigroups Induced by Fuzzy Ideals

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In this paper we define fuzzy quotient semigroups induced by fuzzy ideals and study homomorphism between these fuzzy structures.

1. Introduction

Takashi kuraoka and Nobuaki Kuroki[2] has studied fuzzy quotient ring induced by fuzzy ideals using fuzzy equivalence relations and discussed the relation between fuzzy quotient rings and fuzzy ideals. The aim of this paper is to define fuzzy congruence on a semigroup and to study a fuzzy quotient semigroup using a fuzzy congruence, a fuzzy quotient semigroup induced by fuzzy ideals and also to discuss homomorphism and isomorphism between these fuzzy structures.

2. Preliminaries

Throughout this paper S denotes a semigroup. We recall some definitions and results for the sake of completeness and add some properties of fuzzy ideals.

Definition 2.1. Let X be a nonempty set and μ be a fuzzy relation on X. Then μ is called a fuzzy equivalence relation if

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(i) $\mu(x,x)=1$ for all x in X, (ii) $\mu(x,y)=\mu(y,x)$ for all x and y in X, (iii) for all x and y in X,

$$\mu(x,y) \geq sup_{z\in X} min\{\mu(x,z),\mu(z,y)\} = \mu \circ \mu(x,y)$$

Let μ be a fuzzy equivalence relation on X. We shall say that $\mu[a]$ is the fuzzy class corresponding to a. For each $a \in X$, we denote $\mu[a](x) = \mu(a,x)$ for every $x \in X$. The identity relation Id_X on X is defined for any x, y in X as

$$Id_X(x,y) = \begin{cases} 0 & \text{if } x \neq y \\ 1 & \text{if } x = y \end{cases}$$

Given a fuzzy equivalence relation μ on X, for each α in $[0,1]$, two crisp relations on X are defined as follows.

Definition 2.2. A weak α–relation denoted by ω_α is defined on X as $x\omega_\alpha y$ if and only if $\mu(x,y) \geq \alpha$ and a strong α–relation denoted by σ_α is defined on X as $x\sigma_\alpha y$ if and only if $\mu(x,y) > \alpha$, for any $x, y \in X$.

We call the fuzzy quotient set, the set $X/\mu = \{\mu[a] : a \in X\}$, where μ is a fuzzy equivalence relation on X [2].

Lemma 2.3. *Let μ be a fuzzy equivalence relation on a set X. Then*

(1) $\mu(a,b) = 0$ *if any only if* $min\{\mu[a],\mu[b]\} \equiv 0$,

(2) $sup_{a\in X}\mu[a] \equiv 1$,

(3) $\mu[a] = \mu[b]$ *if and only if* $\mu(a,b) = 1$,

(4) *there exists the surjection* $\rho : X \to X/\mu : x \to \mu[x]$.

Definition 2.4 (1) A fuzzy set δ is a fuzzy semigroup in S if for x, y in $S, \delta(xy) \geq \min\{\delta(x),\delta(y)\}$.

(2) A fuzzy set δ is a fuzzy left, right or two–sided ideal respectively in S if for all $x, y \in S$

$$\delta(xy) \geq \delta(y),$$
$$\delta(xy) \geq \delta(x),$$
$$\delta(xy) \geq max\{\delta(x),\delta(y)\}.$$

Note that the $\cap$ of any set of fuzzy semigroups is a fuzzy semigroup and the $\cap$ or $\cup$ of any set of fuzzy left, right, or two-sided ideals, respectively, is a fuzzy left, right or two-sided ideals. Moreover, η is a fuzzy ideal of S if and only if the level set $\eta_t = \{x \in S | \eta(x) \geq t\}$ is an ideal of S for each $t \in [0, 1]$.

Definition 2.5. Let η be a fuzzy set on a set X and f be a function defined on X. Then η induces a fuzzy set η^f on $f(X)$ defined by

$$\eta^f(y) = sup_{x \in f^{-1}(y)}\{\eta(x)\} \qquad \text{for all } y \in f(X)$$

and is called the image of η under f. Similarly, if ν is a fuzzy set on $f(X)$, then a fuzzy set on X can be defined through $\nu^{f^{-1}}(x) = \nu(f(x)) = \nu \circ f(x)$ for all $x \in X$. where $\circ$ is the composition of mappings. $\nu^{f^{-1}}$ is called the inverse image of ν.

We can prove that a homomorphic image of a fuzzy ideal is a fuzzy ideal without assuming the sup-property.

Lemma 2.6. *Let S and S' be two semigroups and let $f : S \to S'$ be a semigroup homomorphism. If f is surjective and δ is a fuzzy ideal of S, then so is δ^f. If ν is a fuzzy ideal of S', then so is $\nu^{f^{-1}}$.*

Definition 2.7. For a function f from a set X onto the set $f(X)$, a fuzzy set η of X is called f-invariant if $\eta(x) = \eta(y)$ whenever $f(x) = f(y)$ for all x, y in X.

The following theorem gives the correspondence between fuzzy semigroups of S and fuzzy semigroups of homomorphic image of S.

Theorem 2.8. *If $f : S \to f(S)$ is a semigroup homomorphism, then there is a one-to-one correspondence between f-invariant fuzzy semigroups of S and fuzzy semigroups of $f(S)$.*

If $f : S \to S'$ is a semigroup isomorphism (onto) and μ is a fuzzy semigroup of S, then μ is f-invariant. Thus there is a one-to-one correspondence between fuzzy semigroups of S and fuzzy semigroups of S'. This correspondence $\eta \leftrightarrow \eta^f$ is called an isomorphism between fuzzy semigroups η and η^f.

Definition 2.9. Two fuzzy semigroups (S, η) and (S', η') are said to be isomorphic if there exists a mapping $f : S \to S'$ such that f is a semigroup isomorphism and that $\eta' \circ f = \eta$.

3. Fuzzy congruence relation

Now, we define a fuzzy congruence relation on a semigroup S.

Definition 3.1. A fuzzy relation μ on S is said to be left [right] compatible if for any $x, y, a \in S, \mu(ax, ay) \geq \mu(x, y)[\mu(xa, ya) \geq \mu(x, y)]$, and compatible if μ is both left and right compatible. A compatible fuzzy equivalence relation on a semigroup S is called a fuzzy congruence.

Note that the $\cap$ of any fuzzy congruence relation on S is a fuzzy congruence relation.

Theorem 3.2. *If μ is a fuzzy congruence relation on S, then the following are crisp congruence relations on S*

(i) ω_α *for each* $\alpha \in [0, 1]$,

(ii) σ_α *for each* $\alpha \in [0, 1)$.

Proof. (i) Let $0 \leq \alpha \leq 1$. For each $x \in S, \mu(x, x) = 1 \geq \alpha$. Hence $x\omega_\alpha x$. $x\omega_\alpha y$ implies $\mu(x, y) \geq \alpha$ and so $\mu(y, x) = \mu(x, y) \geq \alpha$ implying $y\omega_\alpha x$. Suppose $x\omega_\alpha y$ and $y\omega_\alpha z$. Then $\mu(x, z) = (\mu \circ \mu)(x, z) = sup_t(\mu(x, t) \wedge \mu(t, z)) \geq \mu(x, y) \wedge \mu(y, z) \geq \alpha \wedge \alpha = \alpha$. Finally, if $x\omega_\alpha y$ then $\mu(x, y) \geq \alpha$. Thus, $\mu(ax, ay) \geq \mu(x, y) \geq \alpha$ implies $ax\omega_\alpha ay$. Moreover $\mu(xa, ya) \geq \mu(x, y) \geq \alpha$ implies $xa\omega_\alpha ya$ for any a in S. Similarly (ii) is proved.

Theorem 3.3. *Let S be a semigroup and let μ be a fuzzy congruence on S. Then S/μ is a semigroup under multiplication defined by $\mu[s] * \mu[t] = \mu[st]$ for any s, t in S and $p : S \to S/\nu$ is a homomorphism.*

Proof. By Lemma 2.3, (3), $\mu(x, y) = 1$ if and only if $\mu[x] = \mu[y]$. Let $\mu[x] = \mu[x']$ and $\mu[y] = \mu[y']$. Then $\mu(x, x') = 1$ and $\mu(y, y') = 1$. We claim that $\mu(xy, x'y') = 1$. For $\mu(xy, x'y) \geq \mu(x, x') = 1$ and $\mu(x'y, x'y') \geq \mu(y, y') = 1$. Since $\mu(xy, x'y') \geq (\mu \circ \mu)(xy, x'y') = sup_z\{\mu(xy, z), \mu(z, x'y')\} = 1$. Hence $\mu[xy] = \mu[x'y']$. It is clear that this multiplication $*$ on S/μ is associative. Furthermore, there exists the surjection $p : S \to S/\mu : x \to \mu[x]$. Since $p(x) * p(y) = \mu[x] * \mu[y] =$

$\mu[xy] = p(xy), p$ is homomorphism.

It is useful to observe that if μ is a fuzzy congruence on S, and m and m^* are the multiplications on S and S/μ resp. Then m^* is the unique multiplication on S/μ such that the following diagram commutes

$$\begin{array}{ccccc} S/\mu & \times & S/\mu & \xrightarrow{m^*} & S/\mu \\ p\uparrow & & p\uparrow & & p\uparrow \\ S & \times & S & \xrightarrow{m} & S \end{array}$$

Theorem 3.4. *Let S be a semigroup, δ a fuzzy ideal of S, and $\mu^*(x,y) = (\delta(x) \wedge \delta(y)) \vee Id_S(x,y)$, for all x, y in S. Then μ^* is a fuzzy congruence on S.*

Proof. It is clear that μ is reflexive and symmetric.
Now, if $x = y$, then $(\mu^* \circ \mu^*)(x,y) = 1 = \mu^*(x,y)$, and if $x \neq y$, then

$$\begin{aligned}(\mu^* \circ \mu^*)(x,y) &= sup_z(\mu^*(x,z) \wedge \mu^*(z,y)) \\ &= sup_z(\{(\delta(x) \wedge \delta(z)) \vee Id_S(x,z)\} \wedge \{(\delta(z) \wedge \delta(y)) \vee Id_S(z,y)\}) \\ &\geq \{\delta(x) \wedge \delta(y)\} = \mu^*(x,y).\end{aligned}$$

Thus μ is transitive.

Now we show that μ is compatible. For all a in $S, \mu^*(ax, ay) = (\delta(ax) \wedge \delta(ay)) \vee Id_S(ax, ay)$. If $ax = ay$, then it is clear. If $ax \neq ay$, then

$$\begin{aligned}\mu^*(ax,ay) &= (\delta(ax) \wedge \delta(ay)) \vee Id_S(ax,ay) \\ &\geq (\delta(x) \wedge \delta(y)) = \mu(x,y).\end{aligned}$$

Thus μ is a fuzzy congruence on S.

Theorem 3.5. *Let $f : S \to S'$ be a semigroup (onto) homomorphism. Let δ, η be a fuzzy ideals of S and S', respectively, such that $\delta^f \subset \eta$. Then there is a homormorphism of semigroups $f^* : S/\delta^* \to S'/\eta^*$ such that the* *diagram*

$$\begin{array}{ccc} S & \xrightarrow{f} & S' \\ \downarrow & & \downarrow \\ S/\delta^* & \longrightarrow & S'/\eta^* \end{array}$$

is commutative, where

$$\begin{aligned} \delta^*(x,y) &= (\delta(x) \wedge \delta(y)) \vee Id_S(x,y) \qquad \forall\, x,y \in S, \\ \eta^*(x,y) &= (\eta(x) \wedge \eta(y)) \vee Id_{S'}(x,y) \qquad \forall\, x,y \in S'. \end{aligned}$$

Proof. By Theorem 3.4, δ^* and η^* are congruences on S and S', respectively, and we note that $\delta^*[x_1] = \delta^*[x_2]$ if and only if $\delta^*(x_1, x_2) = 1$. Suppose that $\delta^*[x_1] = \delta^*[x_2]$. If $x_1 = x_2$, then clearly $\eta^*(f(x_1), f(x_2)) = 1$. If $x_1 \neq x_2$, then $\delta(x_1) \wedge \delta(x_2) = 1$ since $\delta^*(x_1, x_2) = 1$. Then $\eta(f(x_1)) \wedge \eta(f(x_2)) \geq \delta(x_1) \wedge \delta(x_2) = 1$ and hence $\eta^*(f(x_1), f(x_2)) = 1$. Thus f^* is well–defined. It is easily seen that f^* is a homomorphism.

Theorem 3.6. *Let $f : S \to S'$ be a homomorphism. Let δ, η be fuzzy ideals of S and S', respectively. Assume $\eta^{f^{-1}} = \delta$. Define $\delta^{**}(x_1, x_2) = (\delta(x_1) \wedge \delta(x_2)) \vee Id_{S'}(f(x_1), f(x_2))$. Then δ^{**} is a fuzzy congruence on S.*

Proof. It is clear that δ^{**} is reflexive and symmetric. Now,

$$(\delta^{**} \circ \delta^{**})(x_1, x_2) = \sup_z(\delta^{**}(x_1, z) \wedge \delta^{**}(z, x_2)).$$

If $x_1 = x_2$, then $(\delta^{**} \circ \delta^{**})(x_1, x_2) = \delta^{**}(x_1, x_2) = 1$. If $x_1 \neq x_2$ and $f(x_1) \neq f(x_2)$, then

$$\begin{aligned} & (\delta^{**} \circ \delta^{**})(x_1, x_2) \\ = \;& \sup_z(\{(\delta(x_1) \wedge \delta(z)) \vee Id_{S'}(f(x_1), f(z))\} \wedge \{(\delta(z) \wedge \delta(x_2))\} \\ & \vee Id_{S'}(f(z), f(x_2))\}) \\ \geq \;& (\delta(x_1) \wedge \delta(x_2)) \vee Id_{S'}(f(x_1), f(x_2)) = \delta^{**}(x_1, x_2). \end{aligned}$$

If $x_1 \neq x_2$ and $f(x_1) = f(x_2)$ then $(\delta^{**} \circ \delta^{**})(x_1, x_2) = \delta^{**}(x_1, x_2) = 1$. Finally,

$$\begin{aligned} \delta^{**}(ax_1, ax_2) &= (\delta(ax_1) \wedge \delta(ax_2)) \vee Id_{S'}(f(ax_1), f(ax_2)) \\ &\geq (\delta(x_1) \wedge \delta(x_2)) \vee Id_{S'}(f(ax_1), f(ax_2)). \end{aligned}$$

If $f(ax_1) = f(ax_2)$, then it is clear that $\delta^{**}(ax_1, ax_2) \geq \delta^{**}(x_1, x_2)$. If $f(ax_1) \neq f(ax_2)$, then

$$\begin{aligned} \delta^{**}(ax_1, ax_2) &= (\delta(ax_1) \wedge \delta(ax_2)) \\ &\geq (\delta(x_1) \wedge \delta(x_2)) = \delta^{**}(x_1, x_2). \end{aligned}$$

Similarly

$$\delta^{**}(x_1 a, x_2 a) \geq \delta^{**}(x_1, x_2).$$

Thus δ^{**} is a fuzzy congruence on S.

Theorem 3.7. *Let $f : S \to S'$ be a onto homomorphism. Let δ, η be fuzzy ideals of S, S', resp. Assume $\eta^{f^{-1}} = \delta$. Then $S/\delta^{**} \cong S'/\eta^*$ holds.*

Proof. By the similar method as in Theorem 3.5, the mapping $f^{**} : S/\delta^{**} \to S/\eta^*$ is onto homomorphism. To show that f^{**} is injective. Suppose $\eta^*[y_1] = \eta^*[y_2]$ for any y_1, y_2 in S'. Since f is onto, there exist x_1 and x_2 in S such that $f(x_1) = y_1$ and $f(x_2) = y_2$. If $y_1 = y_2$ then $\delta^{**}(x_1, x_2) = 1$. Hence $\delta^{**}[x_1] = \delta^{**}[x_2]$. If $y_1 \neq y_2$ then

$$\begin{aligned} \delta^{**}(x_1, x_2) &= (\delta(x_1) \wedge \delta(x_2)) \vee Id_{S'}(f(x_1), f(x_2)) \\ &= \delta(x_1) \wedge \delta(x_2) = \eta(f(x_1)) \wedge \eta(f(x_2)) \\ &= \eta^*(y_1, y_2) = 1. \end{aligned}$$

This completes the proof.

References

[1] I. J. Kumar, P. K. Saxena and Pratibha Yaday, Fuzzy Normal Subgroups and Fuzzy Quotients, *Fuzzy Sets and Systems* Vol. 46 (1992), 121-132.

[2] T. Kuraoka and N. Kuroki, On Fuzzy Quotient Rings Induced by Fuzzy Ideals, *Fuzzy sets and Systems* Vol. 47 (1992), 381-386.

[3] B.T. Lerner, On Fuzzy (Right–Topological) Semigroups, *J. Math. Anal. Appl.* Vol. 134 (1988), 306-311.

[4] V. Murali, Fuzzy Equivalence Relation, *Fuzzy sets and System* Vol. 30 (1989), 155–163.