KYUNGPOOK Math. J. 35(1995), 85-91

Convergence in Frames

Sung Sa Hong
Department of Mathematics, Sogang University, Seoul 121-742, Korea

(1991 AMS Classification number : 06D99 54A99 54D99)

Using covers of a frame, we introduce a concept of convergence of filters in a frame and then characterize compact regular frames by convergence of maximal filters. We also introduce strict extensions of a frame associated with sets of filters in the frame and construct a zero-dimensional compactification of a zero-dimensional frame by the strict extension associated with the set of non-convergent maximal Boolean filters.

0. Introduction

It is well known that the data of convergence of filters in a topological space completely determine the structure in the space and that the theory of frames generalizes that of topological spaces. Frames (= complete Heyting algebras = locales) are also called pointless topological spaces. Although there are no points in frames, there is a possibility to introduce convergence in a frame. The neighborhood filters in a topological space correspond completely prime filters in the frame of the open set lattice of the space, so that one can introduce convergence in a frame using completely prime filters ([6]). In the theory of nearness spaces, one can determine the convergence of filters in a nearness space by its covering structure ([4]).

The purpose of this paper is to introduce a concept of convergence of filters in frames by covers and study its basic properties.

In the first section, we define that a filter F in a frame L is convergent (clustered) if every cover S of L meets F (sec F, resp.). This clearly generalizes convergent filters or filters with cluster points in a topological space. We show that a regular frame L is compact if and only if every

(Received : 16 August 1994)

maximal filter in L is convergent.

The second section concerns strict extensions of frames. Banaschewski defines two extreme cases of extensions of topological spaces i.e., simple and strict ones ([1]). He aslo has a good deal results on compactifications of frames ([2],[3]). For the simple extensions of frames, we refer to ([5], [8]). Using simple extensions and the right adjoints, we introduce a concept of strict extensions of frames. We show that a zero-dimensional frame is compact if and only if every maximal Boolean filter is convergent, where a Boolean filter is a filter generated by its complemented elements. Using this and strict extensions, we construct a zero-dimensional compactification of a zero-dimensional frame.

For the terminology, we mostly refer to [7].

This research is supported by Sogang University Research Grant and we acknowledge its support and we take this opportunity to express our gratitude to Prof. Banaschewski for his suggestions and TGRC for the financial support for Prof. Banaschewski's visit to Korea.

1. Convergence in frames

We first recall that a *frame* is a complete lattice L in which the distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge s : s \in S\}$$

holds for any $a \in$ L and S $\subseteq$ L and that a map between frames is a *frame homomorphism* if it preserves finite meets and arbitrary joins.

In the following, the top and bottom of a frame will be denoted by e and 0, respectively and a filter in a frame always means a proper filter i.e., a filter which does not contain the bottom 0. For a subset F of a frame L, sec F $= \{x \in$ L : for any $a \in$ F, $a \wedge x \neq 0\}$ and for any $x \in$ L, x^* denotes the pseudocomplement of x. By a *cover* of a frame L, we mean a subset S of L with $\bigvee$S $= e$. We say that for A, B $\subseteq$ L, A *refines* B if for any $a \in$ A, there is $b \in$ B with $a \leq b$. It is clear that if a cover A of L refines B, then B is again a cover of L.

We also recall that a filter $\mathcal{F}$ in a nearness space (X, ξ) is convergent if and only if for any member $\mathcal{A}$ of the associated covering structure μ, $\mathcal{F} \cap \mathcal{A} \neq \emptyset$ and that a filter $\mathcal{F}$ in (X, ξ) has a cluster point if and only if for any $\mathcal{A} \in \mu$, sec $\mathcal{F} \cap \mathcal{A} \neq \emptyset$ (see [4] for the details). In particular, a filter $\mathcal{F}$ in a topological space X is convergent if and only if for any open cover $\mathcal{A}$ of X, $\mathcal{F}$ meets $\mathcal{A}$, and it has a cluster point if and only if for any open cover $\mathcal{A}$ of X, sec $\mathcal{F}$ meets $\mathcal{A}$.

Using this, we introduce a concept of convergence in a frame as follows.

Definition 1.1. A filter F in a frame L is said to be

1) *convergent* if for any cover S of L, F meets S,
2) *clustered* if for any cover S of L, sec F meets S.

Remark 1.2. 1) Every completely prime filter in a frame L is convergent and a convergent filter F in L is clustered, for F $\subseteq$ sec F.

2) A filter containing a convergent filter is convergent and a filter contained in a clustered filter is clustered.

3) By 1) and 2), a filter containing a completely prime filter is convergent. If the frame L is Boolean, then every convergent filter is a completely prime filter. Let L be a chain of the interval [0, 1] added with the top e, then the filter $\{1, e\}$ is convergent but not a completely prime filter.

4) A maximal filter in a frame L is convergent if and only if it is clustered, because sec F = F for a maximal filter F.

5) Let B be a base for a frame L, i.e., for any $x \in$ L, there is a subset C of B with $x = \bigvee$C. Then a filter F is convergent (clustered) if and only if for any C $\subseteq$ B with $\bigvee$C $= e$, F (sec F, resp.) meets C.

Proposition 1.3. *A filter F in a frame L is clustered if and only if* $\bigvee\{x^* : x \in F\} \neq e$.

Proof. Suppose that $\bigvee\{x^* : x \in \text{F}\} = e$, then by the assumption, there is an $x \in$ F with $x^* \in$ sec F, which is a contradiction. For the converse, assume that there is a cover S of L such that sec F $\cap$ S $= \emptyset$, then for any s $\in$ S, there is $x_s \in$ F with $s \wedge x_s = 0$; hence $s \leq x_s^*$, so that S refines $\{x^* : x \in \text{F}\}$, which is again a contradiction.

We recall that a frame L is almost compact if for any cover S of L, there is a finite F $\subseteq$ S with $(\vee F)^* = 0$. It is known that a frame L is almost compact if and only if for any filter F in L, $\bigvee\{x^* : x \in \text{F}\} \neq e$ and that a regular frame is almost compact if and only if it is compact ([5],

[8]). Thus the following are immediate from the above proposition.

Corollary 1.4. *For a frame L, the following are equivalent:*

1) *L is almost compact.*
2) *Every filter in L is clustered.*
3) *Every maximal filter in L is convergent.*

Corollary 1.5. *For a regular frame L, the following are equivalent:*

1) *L is compact.*
2) *Every filter in L is clustered.*
3) *Every maximal filter in L is convergent.*

Let $f : \mathrm{L} \longrightarrow \mathrm{M}$ be a frame homomorphism. Then for any filter F in M, $f^{-1}(\mathrm{F})$ is again a filter in L. Moreover, if f is dense, i.e., $f(a) = 0$ implies $a = 0$, then for any filter F in L, $f(\mathrm{F})$ is a filter base in M. Furthermore, if f is dense onto, then $f(\mathrm{F})$ is a filter in M. We recall that a frame homomorphism $f : \mathrm{L} \longrightarrow \mathrm{M}$ is *codense* if $f(a) = e$ implies $a = e$.

Proposition 1.6 *Let $f : L \longrightarrow M$ be a frame homomorphism.*

1) *If F is a filter in M which is convergent (clustered), then $f^{-1}(F)$ is also convergent (clustered, resp.) in L.*

2) *Assume that f is dense, codense and onto, and a filter F in L is convergent (clustered), then $f(F)$ is also convergent (clustered, resp.) in M.*

Proof. 1) Take any cover S of L, then $f(\mathrm{S})$ is clearly a cover of M; hence $\mathrm{F} \cap f(\mathrm{S}) \neq \emptyset$ (sec $\mathrm{F} \cap f(\mathrm{S}) \neq \emptyset$, resp.). Pick $s \in \mathrm{S}$ with $f(s) \in \mathrm{F}$ ($f(s) \in$ sec F, resp.), then clearly $s \in f^{-1}(\mathrm{F}) \cap \mathrm{S}$ ($s \in$ sec $f^{-1}(\mathrm{F}) \cap \mathrm{S}$, resp.).

2) Suppose S is a cover of M, then $f^{-1}(\mathrm{S})$ is again a cover of L, for f is onto, codense. Therefore, there is $t \in f^{-1}(\mathrm{S}) \cap \mathrm{F}$ ($t \in f^{-1}(\mathrm{S}) \cap$ sec F, resp.), which implies $f(t) \in \mathrm{S} \cap f(\mathrm{F})$ ($f(t) \in \mathrm{S} \cap$ sec $f(\mathrm{F})$, resp., because f is dense).

2. Strict extensions of frames

In this section, we introduce a concept of strict extensions of frames and then we construct a zero-dimensional compactification of a zero-dimensional frame.

In what follows, X denotes a set of filters in a frame L and P(X) the frame of the power set lattice. Furthermore, we let

$$s_X L = \{(x, \Sigma) \in L \times P(X) : \text{ for any } F \in \Sigma, x \in F\}$$

and let $s : s_X\mathrm{L} \longrightarrow \mathrm{L}$ be the restriction of the first projection to $s_X\mathrm{L}$. Then $s_X\mathrm{L}$ is a subframe of the product frame of L and P(X) and s is an open dense onto frame homomorphism, which is called the *simple extension* of L with respect to X (see [5], [8] for the detail).

Let s^* denote the right adjoint of s, then $s^*(x) = (x, \Sigma_x)$ for any $x \in \mathrm{L}$, where $\Sigma_x = \{\mathrm{F} \in \mathrm{X} : x \in \mathrm{F}\}$. Clearly, $s^*(\mathrm{L})$ is closed under finite meets in $s_X\mathrm{L}$ and let $t_X\mathrm{L}$ be the subframe of $s_X\mathrm{L}$ generated by $s^*(\mathrm{L})$. Then $t_X\mathrm{L} = \{\bigvee\{(x, \Sigma_x) : x \in \mathrm{A}\} : \mathrm{A} \subseteq \mathrm{L}\}$. Let $t : t_X\mathrm{L} \longrightarrow \mathrm{L}$ be the restriction of s to $t_X\mathrm{L}$, which is clearly a dense onto frame homomorphism.

Using the above notation, we now define the following.

Definition 2.1. The frame homomorphism $t : t_X\mathrm{L} \longrightarrow \mathrm{L}$ or $t_X\mathrm{L}$ is called the *strict extension* of L with respect to X.

Remark 2.2. Let L be a frame $\Omega(\mathrm{E})$ of the open set lattice of a topological space E and $\{\mathrm{T}_y : y \in \mathrm{E}'\}$ a family of open filters in E which extends the family of open neighborhood filters of E. Let $\mathrm{X} = \{\mathrm{T}_y : y \in \mathrm{E}' - E\}$, then $t_X\mathrm{L}$ is precisely the open set frame of the strict extension of the space E with respect to $\{\mathrm{T}_y : y \in \mathrm{E}'\}$ in the sense of Banaschewski ([1]).

In the following, let C(L) denote the set of complemented elements of a frame L and we recall that a frame L is *zero-dimensional* if C(L) is a base for L. For any $x \in \mathrm{C(L)}$, x' denotes the complement of x.

Definition 2.3. A filter F in a frame is said to be *Boolean* if it is generated by $\mathrm{F} \cap \mathrm{C(L)}$, i.e., for any $x \in \mathrm{F}$, there is a complemented element $y \in \mathrm{F}$ with $y \leq x$. By a *maximal Boolean filter*, we mean a Boolean filter which is maximal in the set of Boolean filters in L with the inclusion.

We note that a filter on a topological space E is a clopen filter if and only if it is a Boolean filter in $\Omega(\mathrm{E})$.

Proposition 2.4. *For a zero-dimensional frame L, the following are equivalent:*

1) *L is compact.*

2) *Every Boolean filter in L is clustered.*

3) *Every maximal Boolean filter in L is convergent.*

Proof. 1) $\Rightarrow$ 2). It is immediate from Corollary 1.4.

2) $\Rightarrow$ 3). We note that a Boolean filter in L is maximal if and only if sec F $\cap$ C(L) $\subseteq$ F. Thus the implication follows from 5) of Remark 1.2, for C(L) is a base for L.

3) $\Rightarrow$ 1). Suppose that there is a cover S of L which does not have a finite subcover. Let T $= \{t \in$ C(L) $: t \leq s$ for some $s \in$ S$\}$, then T is a cover of L, which does not have a finite subcover. Thus $\{x' : x \in$ T$\}$ generates a Boolean filter, which is denoted by F. Let G be a maximal Boolean filter containing F. By the assumption, G is convergent, so that G $\cap$ T $\neq \emptyset$. Pick $t \in G \cap$ T, then $t, t' \in$ G, which is a contradiction.

In the remainder of the section, L is always a zero-dimensional frame and X is the set {F : F is a non-convergent maximal Boolean filter}. Furthermore, $s^*(\mathrm{C(L)}) = \{(x, \Sigma_x) \in t_X \mathrm{L} : x \in \mathrm{C(L)}\}$.

Lemma 2.5. *$s^*(C(L))$ is contained in $C(t_X L)$ and is closed under finite meets in $t_X L$. Furthermore, $s^*(C(L))$ generates $t_X L$.*

Proof. The first part follows from the fact that for any maximal Boolean filter F and any $x \in$ C(L), $x \in$ F or $x' \in$ F, and the second half is trivial, for C(L) is closed under finite meets in L. We note that for any $a \in$ L, $(a, \Sigma_a) = \bigvee\{(x, \Sigma_x) : x \in \mathrm{C(L)} \cap \downarrow a\}$, because L is zero-dimensional and X consists of Boolean filters. Thus t_XL is generated by s^*(C(L)).

Notation 2.6. The extension $t : t_X \mathrm{L} \longrightarrow \mathrm{L}$ will be denoted by $\zeta : \zeta L \rightarrow L$.

Using this notion, we have the following:

Theorem 2.7. *ζL is a zero-dimensional compact frame and hence $\zeta : \zeta L \longrightarrow L$ is a zero-dimensional compactification of L.*

Proof. It follows from the above lemma that ζL is zero-dimensional. It remains to show that it is compact. Take any maximal Boolean filter Ψ in ζL, then $\zeta(\Psi)$ is also a Boolean filter, for ζ is a dense, onto frame homomorphism and $\zeta(\mathrm{C}(\zeta\mathrm{L})) \subseteq \mathrm{C(L)}$. It is easy to show that s^*(sec $\zeta(\Psi)$

$\cap$ C(L)) $\subseteq$ C(ζL) $\cap$ sec $\Psi \subseteq \Psi$; hence $\zeta(\Psi)$ is a maximal Boolean filter in L. Suppose that $\zeta(\Psi)$ is convergent in L. Let $\Phi = \{\alpha \in \zeta L : \beta \leq \beta$ for some $\beta \in \zeta^{-1}(\zeta(\Psi)) \cap$ C(ζL)$\}$, then it is a Boolean filter in ζL containing Ψ. Therefore, $\Psi = \Phi$. By Proposition 1.6, $\zeta^{-1}(\zeta(\Psi))$ is convergent and since ζL is zero-dimensional, $\Psi = \Phi$ is also convergent in ζL. Now suppose that $\zeta(\Psi)$ is not convergent, then $\zeta(\Psi) \in$ X. Take any cover S of ζL with S $\subseteq s^*$(C(L)) which is a base for ζL. Let $p : \zeta$L $\longrightarrow$ P(X) be the restriction of the second projection of L $\times$ P(X), which is a frame homomorphism. Thus p(S) is a cover of P(X); $\zeta(\Psi) \in p((x, \Sigma_x)) = \Sigma_x$ for some $(x, \Sigma_x) \in$ S. Hence there is $\alpha \in \Psi$ with $\zeta(\alpha) = x$, so that $\alpha \leq \zeta^*(x) = (x, \Sigma_x)$. Thus $(x, \Sigma_x) \in$ S $\cap$ Ψ; therefore Ψ is convergent in ζL. This completes the proof.

References

[1] B. Banaschewski, Extensions of topological spaces, Can. Math. Bull., 7(1964), 1-22.

[2] B. Banaschewski, Frames and compactifications, Extension Theory of Topological Structures and its Appl., Deutscher Verlag der Wissenschaften, Berlin, 1969, 29-33.

[3] B. Banaschewski, Compactification of frames, Math. Nachr. 149(1990), 105-116.

[4] H. Herrlich, Topological Structures, Math. Centre Tracts, 52(1974), 59-122.

[5] S. S. Hong, Simple extensions of frames, Proc. Recent Devel. of Gen. Top. and its Appl., Math. Research, 67(1992), 156-159, Akademia Verlag, Berlin.

[6] J. R. Isbell, Atomless parts of spaces, Math. Scand. 31(1972), 5-32.

[7] P. T. Johnstone, Stone Spaces, Cambridge Studies in Advanced Math., 3, Cambridge Univ. Press, 1982

[8] J. Paseka and B. Šmarda, T_2-frames and almost compact frames, Czech. Math. J., 42(1992), 385-402.