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Harmonic Conjugates of Bloch Functions on Half-Spaces

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For a given harmonic Bloch function u vanishing at some point z_0 on the upper half-space, we represent unique harmonic conjugates of u which are also Bloch functions vanishing at z_0 in terms of partial derivatives of u.

The upper half-space $H = H_n$ is the open subset of $\mathbf{R}^n (n \geq 2)$ given by

$$H = \{(x, y) \in \mathbf{R}^n : y > 0\},$$

where we have written a typical point $z \in \mathbf{R}^n$ as $z = (x, y)$, with $x \in \mathbf{R}^{n-1}$ and $y \in \mathbf{R}$.

Given a harmonic function u on H, the functions $v_1, \cdots, v_{n-1}$ on H are called harmonic conjugates of u if

$$(v_1, \cdots, v_{n-1}, u) = \nabla f \tag{1}$$

for some harmonic function u on H, where ∇f denotes the gradient of f. If (1) holds, then $v_1, \cdots, v_{n-1}$ are partial derivates of a harmonic function, so they are harmonic on H. Also (1) and the condition that f is harmonic is equivalent to the following "generalized Cauchy-Riemann equations"

$$D_k v_j = D_j v_k; D_n v_j = D_j u$$

$$\sum_{j=1}^{n-1} D_j v_j + D_n u = 0.$$

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In particular, v is a harmonic conjugate of u if and only if $u + iv$ is holomorphic on the upper half-plane H_2.

If u is harmonic on H, then harmonic conjugates of u always exists. Unfortunately, they are far from unique. (When $n > 2$, harmonic conjugates for a given u may well differ by more than a constant. We refer more on harmonic conjugates to [AR], [S] and [SW].)

In this paper, we are interested in harmonic conjugates of Bloch functions on H. Recall that a harmonic function u on H is called a Bloch function if

$$\|u\|_{\mathcal{B}} = \sup y|\nabla u(x,y)| < \infty,$$

where the supremum is taken over all $(x, y) \in H$. (Here we use the $\mathbf{C}^n$-norm to calculate $|\nabla u(x,y)|$.) We let $\mathcal{B}$ denote the collection of Bloch functions on H and let $\widetilde{\mathcal{B}}$ denote the subspace of functions in $\mathcal{B}$ that vanish at $z_0 = (0,1)$. Then we can show easily that $\widetilde{\mathcal{B}}$ is a Banach space under the Bloch norm $\| \ \|_{\mathcal{B}}$.

Below we show that if $u \in \widetilde{\mathcal{B}}$, we can choose harmonic conjugates of u which can be written in terms of its partial derivatives and we show these conjugates are unique conjugates belonging to $\widetilde{\mathcal{B}}$. For this purpose, we first let

$$R(z,w) = \frac{4}{nV(B)} \frac{n(z_n + w_n)^2 - |z - \bar{w}|^2}{|z - \bar{w}|^{n+2}}$$

for $z = (z_1, \cdots, z_{n-1}, z_n), w = (w_1, \cdots, w_{n-1}, w_n) \in H$, where $V(B)$ denotes the volume of the unit ball B in $\mathbf{R}^n$ and $\bar{w} = (w_1, \cdots, w_{n-1}, -w_n)$. (Note that if $n = 2$, then $\bar{w}$ is the usual complex conjugate of w.) Then it is shown in [RY] that the function $\widetilde{R}$ defined on $H \times H$ by

$$\widetilde{R}(z,w) = R(z,w) - R(z_0,w)$$

has the following reproducing properties: If $u \in \widetilde{\mathcal{B}}$, then

$$u(z) = \int_H u(w)\widetilde{R}(z,w)dw = -2\int_H u(w)w_n D_{w_n}\widetilde{R}(z,w)dw \tag{2}$$

for $z \in H$. (Here $dw = dV(w)$ denotes the Lebesgue volume measure in $\mathbf{R}^n$.) Furthermore from the definition of $\widetilde{R}$, we can show that for each $j = 1, \cdots, n$, there is a constant C depending only on n and z such that

$$|\widetilde{R}(z,w)|, |w_n D_{w_j}\widetilde{R}(z,w)| \le \frac{C(n,z)}{1 + |w|^{n+1}} \tag{3}$$

for all $w \in H$. Now fix $u \in \widetilde{\mathcal{B}}$ for the rest of this paper and let

$$v_j(z) = -2 \int_H [D_{w_j} u(w)] w_n \widetilde{R}(z,w) dw$$

for $z \in H$. Then from the first estimate of (3), we have

$$|[D_{w_j} u(w)] w_n \widetilde{R}(z,w)| \leq C(n,z) \frac{\|u\|_{\mathcal{B}}}{1+|w|^{n+1}},$$

which belongs to $L^1(H)$ as a function of w. Therefore v_j makes sense. Moreover, $v_j(z_0) = 0$ and by passing the Laplacian Δ_z through the integral above, we easily see that v_j is a harmonic function on H. Note that for $k = 1, 2, \cdots, n$,

$$\begin{aligned}
|z_n D_{z_k} v_j(z)| &= 2|z_n \int_H [D_{w_j} u(w)] w_n D_{z_k} \widetilde{R}(z,w) dw| \\
&= 2|z_n \int_H [D_{w_j} u(w)] w_n D_{z_k} R(z,w) dw| \\
&\leq 2\|u\|_{\mathcal{B}} z_n \int_H \frac{C(n)}{|z-\bar{w}|^{n+1}} dw \\
&\leq 2C(n)\|u\|_{\mathcal{B}} z_n \int_0^\infty \frac{1}{(z_n+w_n)^2} \Big(\int_{R^{n-1}} \frac{z_n+w_n}{|z-\bar{w}|^n} \\
&\quad dw_1 \cdots dw_{n-1} \Big) dw_n.
\end{aligned}$$

The inner integral above equals

$$\frac{nV(B)}{2} \int_{\mathbf{R}^{n-1}} P(z+(0,w_n),(w_1,\cdots,w_{n-1},0)) dw_1 \cdots dw_{n-1} = \frac{nV(B)}{2},$$

where P is the Poisson kernel for the upper half space. (We refer more on Poisson kernel to [ABR].) Hence

$$\begin{aligned}
|z_n D_{z_k} v_j(z)| &\leq C(n)\|u\|_{\mathcal{B}} z_n \int_0^\infty \frac{1}{(z_n+w_n)^2} dw_n \\
&\leq C(n)\|u\|_{\mathcal{B}}.
\end{aligned}$$

(Here $C(n)$ denotes a constant depending on n whose value may change from line to line.) Therefore $v_j \in \widetilde{\mathcal{B}}$ and $\|v_j\|_{\mathcal{B}} \leq C(n)\|u\|_{\mathcal{B}}$. To get the main result, we need one lemma whose proof relies on integration by parts.

(4) **Lemma**. *For $j = 1, 2, \cdots, n-1$ and for $z \in H$,*

$$v_j(z) = 2\int_H u(w) w_n D_{w_j} \widetilde{R}(z,w)\,dw \tag{5}$$

Proof. First note that the integral in (5) makes sense. We can easily see theis from (3) and the following estimate;

$$|u(w)| \le 2\|u\|_{\mathcal{B}}(1 + |\log w_n| + 2\log(1+|w|)) \tag{6}$$

for $w \in H$. (See [AR].) Thus the right side of (5) equals

$$2\int_{R^{n-2}}\int_0^\infty w_n \int_{-\infty}^\infty u(w) D_{w_j}\widetilde{R}(z,w)\,dw_j\,dw_n\,\widehat{dw_{jn}}, \tag{7}$$

where $\widehat{dw_{jn}} = dw_1 \cdots dw_{j-1} dw_{j+1} \cdots dw_{n-1}$. From estimates (3) and (6) we can also show that $|u(w)\widetilde{R}(z,w)| \to 0$ as $|w_j| \to \infty$. Now integrating by parts in the innermost integral above, (7) becomes

$$\begin{aligned} & -2\int_{R^{n-2}}\int_0^\infty w_n \int_{-\infty}^\infty [D_{w_j} u(w)]\widetilde{R}(z,w)\,dw_j\,dw_n\,\widehat{dw_{jn}} \\ = & -2\int_H [D_{w_j} u(w)] w_n \widetilde{R}(z,w)\,dw \\ = & \; v_j(z). \end{aligned}$$

This completes the proof.

(8) **Theorem**. *The functions $v_1, \cdots, v_{n-1}$ defined above are unique harmonic conjugates of u belonging to $\widetilde{\mathcal{B}}$.*

Proof. We know each $v_j \in \widetilde{\mathcal{B}}$. To show $v_1, \cdots, v_{n-1}$ are harmonic conjugates of u, note that for $j, k = 1, 2, \cdots, n-1$,

$$D_{z_k} D_{w_j}\widetilde{R}(z,w) = -D_{z_j} D_{z_k} R(z,w) = D_{z_j} D_{w_k}\widetilde{R}(z,w),$$

$$D_{z_n} D_{w_j}\widetilde{R}(z,w) = -D_{z_j} D_{w_n}\widetilde{R}(z,w). \tag{9}$$

Note also that

$$D_{z_n} D_{w_n} \widetilde{R}(z,w) = D^2_{z_n} R(z,w).$$

Hence by differentiating through the integral in (5), we have $D_k v_j = D_j v_k$ for $j,k = 1,2,\cdots,n-1$. Similary from (2) and (9), we get $D_n v_j = D_j u$. Finally,

$$(\sum_{j=1}^{n-1} D_j v_j + D_n u)(z) = -2\int_H u(w) w_n \Delta_z R(z,w)dw \equiv 0$$

for all $z \in H$.
Hence $v_1,\cdots,v_{n-1},u$ satisfy the generalized Cauchy-Riemann equations and it follows that $v_1,\cdots,v_{n-1}$ are harmonic conjugates of u.

To complete the proof, suppose that $u_1,\cdots,u_{n-1}$ are also harmonic conjugates of u such that $u_j \in \widetilde{\mathcal{B}}$ for each j. Then

$$\|v_j - u_j\|_{\mathcal{B}} \le C(n)\|z_n D_{z_n}(v_j - u_j)\|_\infty.$$

(See Theroem 5.13 of [RY].) Since $D_{z_n}(v_j - u_j) = D_j(u-u) = 0$, we get $\|v_j - u_j\|_{\mathcal{B}} = 0$ and so $v_j = u_j$ as desired.

Remark. Given a harmonic Bloch function u on the upper half space, the existence of unique harmonic conjugates of u which are also Bloch, was shown in [AR] for the first time using a normal family argument.

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