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On Approximation by a Power Series Type Operator

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In the present paper we study some quantitative estimates on a sequence of power series operators introduced by J. Swetits and B. Wood by using Peetre K - functional and modulus of continuity.

1. Introduction

Meir and Sharma [3] introduced a generalization of the S_α method of the summability by a matrix $(a_{n,k})$ defined as

$$\prod_{j=0}^{n} \frac{(1-\alpha_j)}{(1-\alpha_j\theta)} = \sum_{k=0}^{\infty} a_{n,k}\theta^k \tag{1.1}$$

where $\{\alpha_j\}_{j=0}^{\infty}$ is a sequence of complex numbers and $0 < \theta < 1$.

If $0 < \alpha_j < 1$ (for $j = 0, 1, 2, \cdots$), then $a_{n,k} \geq 0$ (for $n = 0, 1, 2, \cdots$ and $k = 0, 1, 2, \cdots$). By putting

$$\alpha_j = \alpha_j(x) = \frac{h_j(x)}{1+h_j(x)}; j = 0, 1, 2, \cdots,$$

in the result (1.1), the following identity

$$R_n(x;\theta) = \prod_{j=0}^{n} \frac{1}{1+h_j(x)-h_j(x)\theta} = \sum_{k=0}^{\infty} C_{n,k}(x)\theta^k \tag{1.2}$$

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is obtained where $\{h_j(x)\}_{j=0}^{\infty}$ is a sequence of non-negative real valued functions defined on $[0, \infty)$.

Swetits and Wood [6] studied the following positive linear operator $\{L_n\}$ of order n defined as

$$(L_n f)(x) = \sum_{k=0}^{\infty} C_{n,k}(x) f(\frac{k}{n}) \text{ for } f \in C[0, \infty) \tag{1.3}$$

and proved the following result on convergence.

Theorem. *Suppose $\{h_j(x)\}$ is a sequence of continuous non-negative real valued functions defined on $[0, \infty)$. Suppose that on each interval $[0, a]$ there is a constant M which depends only on a such that $h_j(x) \leq M$ for $j = 0, 1, 2, \cdots$, $x \in [0, a]$. Let f be continuous on $[0, \infty)$ and satisfy $|f(x)| \leq e^{Ax}$ for some constant $A > 0$. Then the sequence $\{L_n f\}_{n=0}^{\infty}$ defined in (1.3) converges to f uniformly on $[0, a]$ if $\{h_j\}_{j=0}^{\infty}$ is uniformly $(c, 1)$ summable to x on $[0, a]$.*

By putting $h_j(x) = x, \forall j$ in the operator (1.3), the following (Baskakov [1])

$$(V_n f)(x) = (1 + x)^{-n} \sum_{k=0}^{\infty} \binom{n + k - 1}{k} (\frac{x}{1 + x})^k f(\frac{k}{n}), \quad f \in C[0, \infty) \tag{1.4}$$

is obtained.

In the present paper [5] we prove some quantitative theorems on the operator (1.3).

2. Results

Proposition 2.1. *For $n \geq 1$, the following moments are obtained*

$$(L_n 1)(x) = 1 \tag{2.1}$$

$$(L_n t)(x) = \frac{1}{n}[\frac{\partial}{\partial \theta} R_n(x, 1)] \tag{2.2}$$

$$(L_n t^2)(x) = \frac{1}{n^2}[\frac{\partial}{\partial \theta} R_n(x, 1) + \frac{\partial^2}{\partial \theta^2} R_n(x, 1)] \tag{2.3}$$

In particular, one gets

$$\begin{aligned}(L_n(t-x)^2)(x) &= \frac{1}{n^2}[\frac{\partial^2}{\partial\theta^2}R_n(x,1)+(1-2nx)\frac{\partial}{\partial\theta}R_n(x,1)]+x^2 \\ &= H_n(x) \text{ (say)} \qquad (2.4)\end{aligned}$$

Proof. By putting $e_i(x) = x^i$ for $i = 0, 1, 2$ in the operator (1.3), the results (2.1) to (2.4) follow easily.

Theorem 2.2. *If $g \in C_B^{(2)}[0,\infty)$ then for $n \geq 1$, we have the following estimate,*

$$|(L_n g)(x) - g(x)| \leq M_n(x)\|g\|_{C_B^{(2)}} \qquad (2.5)$$

where

$$M_n(x) = \max\{|\frac{1}{n}\frac{\partial}{\partial\theta}R_n(x,1) - x|; \frac{1}{2}|H_n(x)|\} \qquad (2.6)$$

and

$$\|g\|_{C_B^{(2)}} = \|g\|_{C_B} + \|g'\|_{C_B} + \|g''\|_{C_B} \qquad (2.7)$$

for a bounded and uniformly continuous function f on the interval $[0,\infty)$ with the norm $\|\cdot\|$ defined as

$$\|f\|_{C_B} = \sup_{0\leq t<\infty} |f(t)|.$$

Proof. By applying the Taylor's expansion for $g \in C_B^{(2)}[0,\infty)$, we write that

$$g(t) - g(x) = (t-x)g'(x) + \frac{1}{2}(t-x)^2 g''(\xi)$$

where $\min(x,t) \leq \xi \leq \max(x,t)$. Using the expression (1.3) and the results (2.1) to (2.4), we see that

$$\begin{aligned}|(L_n g)(x) - g(x)| &\leq \|g'\||L_n(t-x)(x)| + \frac{1}{2}\|g''\||L_n(t-x)^2(x)| \\ &\leq \|g'\||\frac{1}{n}\frac{\partial}{\partial\theta}R_n(x,1) - x| + \frac{1}{2}\|g''\||H_n(x)| \\ &\leq M_n(x)\{\|g'\| + \|g''\|\} \\ &\leq M_n(x)\|g\|_{C_B^{(2)}}\end{aligned}$$

This completes the proof.

For a function $f \in C_B[0,\infty)$ the Peetre K-functional is defined as ([4])

$$k(f;u) = \inf_{g\in C_B^{(2)}} \{\|f-g\|_{C_B} + u\|g\|_{C_B^{(2)}}\}$$

where $u \geq 0$ is any real number. The Peetre K-functional is related to second order of modulus of continuity as below

$$k(f;u) \leq A\{\omega_2(f;\sqrt{u}) + \min(1,u)\|f\|_{C_B}\} \tag{2.8}$$

where the constant A depends only on f and u.

Theroem 2.3. *For $f \in C_B[0,\infty)$, one gets that,*

$$|(L_nf)(x) - f(x)| \leq 2A\{\omega_2(f : \sqrt{\frac{M_n(x)}{2}}) + \min(1; \frac{M_n(x)}{2})\|f\|_{C_B}\} \tag{2.9}$$

where $M_n(x)$ and A are already defined.

Proof. For $f \in C_B[0,\infty)$ and $g \in C_B^{(2)}[0,\infty)$ we can write that

$$\begin{aligned} |(L_nf)(x) - f(x)| &\leq |(L_nf)(x) - (L_ng)(x)| + |(L_ng)(x) - g(x)| \\ &\quad + |f(x) - g(x)| \\ &\leq \|L_n\|\|f-g\|_{C_B} + M_n(x)\|g\|_{C_B^{(2)}} + \|f-g\|_{C_B} \\ &\leq 2\{\|f-g\|_{C_B} + \frac{1}{2}M_n(x)\|g\|_{C_B^{(2)}}\}. \end{aligned}$$

Taking infimum over $g \in C_B^{(2)}[0,\infty)$ and using the result (2.8), we see that

$$\begin{aligned} |(L_nf)(x) - f(x)| &\leq 2k(f;\frac{1}{2}M_n(x)) \\ &\leq 2A\{\omega_2(f;\sqrt{\frac{M_n(x)}{2}}) + \min(1;\frac{M_n(x)}{2})\|f\|_{C_B}\} \end{aligned}$$

This completes the proof.

Theorem 2.4. *Let $f \in C^{(1)}[0,a], a > 0$. Then for $n \geq 1$*

$$\begin{aligned} |(L_n f)(x) - f(x)| &\leq \|f'\| |\frac{1}{n}\frac{\partial}{\partial\theta} R_n(x,1) - x| \\ &+\frac{3}{2}\omega(f'; \sqrt{H_n(x)})\sqrt{H_n(x)} \end{aligned} \tag{2.10}$$

where $\omega(f';\cdot)$ is the modulus of continuity of f' and $H_n(x)$ is defined in the result (2.4).

Proof. We have

$$f(t) - f(x) = (t-x)f'(x) + \int_0^1 (t-x)[f'\{x+\theta(t-x)\} - f'(x)]d\theta$$

so applying (1.3) we get that

$$\begin{aligned} &|(L_n f)(x) - f(x)| \\ \leq\ & |f'(x)||L_n(t-x)(x)| + [L_n(t-x)\int_0^1 \{f'(x+\theta(t-x)) - f'(x)\}d\theta](x) \\ \leq\ & \|f'\||L_n(t-x)(x)| + [L_n|t-x|\int_0^1 |f'(x+\theta(t-x)) - f'(x)|d\theta](x) \\ \leq\ & \|f'\||L_n(t-x)(x)| + [L_n|t-x|\int_0^1 \omega(f';|\theta||t-x|)d\theta](x) \\ \leq\ & \|f'\||L_n(t-x)(x)| + [L_n|t-x|\int_0^1 (1+\frac{|\theta||t-x|}{\delta})\omega(f';\delta)d\theta](x) \\ \leq\ & \|f'\||L_n(t-x)(x)| + \omega(f',\delta)[L_n|t-x|(1+\frac{|t-x|}{2\delta})](x) \\ \leq\ & \|f'\||L_n(t-x)(x)| + \omega(f',\delta)[L_n|t-x|(x) + \frac{1}{2\delta}(L_n(t-x)^2)(x)] \end{aligned}$$

Further using (2.1) to (2.4) and schwarz inequality we get that

$$|(L_n f)(x) - f(x)| \leq \|f'\||\frac{1}{n}\frac{\partial}{\partial\theta}R_n(x,1) - x| + \omega(f';\delta)[\sqrt{H_n}(x) + \frac{1}{2\delta}H_n(x)].$$

Finally choosing $\delta = \sqrt{H_n(x)}$, we get the required result. This completes the proof.

Theorem 2.5. *For $f \in C^{(1)}[0,a], a > 0$, let $(L_n f)$ be a sequence of positive linear operators defined in (1.3). Then for $n \geq 1$ the following estimates hold*

$$\begin{aligned} |(L_n f)(x) - f(x)| &\leq (3 + \frac{H_n(x)}{\delta^2})\omega_2(f;\delta) \\ &\quad + \frac{2}{\delta}|\frac{1}{n}\frac{\partial}{\partial\theta}R_n(x,1) - x|\omega_1(f;\delta) \qquad (2.11) \\ &\leq \omega_1^*(f;\sqrt{H_n(x)}) \qquad (2.12) \\ &\leq \|f'\||\frac{1}{n}\frac{\partial}{\partial\theta}R_n(x,1) - x| \\ &+ \omega_1(f';\delta)[\sqrt{H_n(x)} + \frac{1}{2\delta}H_n(x)] \qquad (2.13) \end{aligned}$$

Where, $\omega_1^*(f;\cdot)$ is the least concave majorant of $\omega_1(f;\cdot)$ and (2.13) is valid for $f \in C^{(1)}[a,b]$.

Proof. The proof follows from ([2] Theorem 2.1) on using (2.1) to (2.4).

3. A Special Case

Baskakov Operator. By putting $h_j(x) = x \, \forall j$ in the result (2.1) to (2.4) we get that

$$(L_n 1)(x) = 1 \qquad (3.1)$$

$$(L_n t)(x) = x \qquad (3.2)$$

$$(L_n t^2)(x) = x^2(1 + \frac{1}{n}) + \frac{x}{n} \qquad (3.3)$$

In particular,

$$L_n(t - x)(x) = 0 \qquad (3.4)$$

and

$$L_n(t-x)^2(x) = \frac{x(1+x)}{n} = H_n(x) \text{ (say)} \qquad (3.5)$$

and also

$$M_n = \max\{0; \frac{1}{2}|\frac{x(1+x)}{n}|\} = \frac{x(1+x)}{2n} \qquad (3.6)$$

We get the following results on Baskakov operators (1.4).

Theorem. *If $g \in C_B^{(2)}[0,\infty)$ then for $n \geq 1$,*

$$|(V_n g)(x) - g(x)| \leq (\frac{x(1+x)}{2n})\|g\|_{C_B^{(2)}} \tag{3.7}$$

Theorem. *For $f \in C_B[0,\infty)$ then for $n \geq 1$*

$$|(V_n f)(x) - f(x)| \leq 2A[\omega_2(f; \frac{1}{2}\sqrt{\frac{x(1+x)}{n}}) + \min(1; \frac{x(1+x)}{4n})\|f\|_{C_B}] \tag{3.8}$$

Theorem. *Let $f \in C^{(1)}[0,\infty)$ and let $\omega(f';\cdot)$ be the modulus of continuity of f'. Then for $n \geq 1$ we have*

$$|(V_n f)(x) - f(x)| \leq \frac{3}{2}\omega(f; \frac{1}{2}\sqrt{\frac{x(1+x)}{n}})\sqrt{\frac{x(1+x)}{n}} \tag{3.9}$$

Theorem. *Under the assumptions, the Theorem 2.5 reduces to the following result*

$$|(V_n f)(x) - f(x)| \leq \{3 + \frac{x(1+x)}{\delta^2 n}\}\omega_2(f;\delta) \tag{3.10}$$

$$|(V_n f)(x) - f(x)| \leq \omega_1^*(f; \sqrt{\frac{x(1+x)}{n}}) \tag{3.11}$$

$$|(V_n f)(x) - f(x)| \leq [\sqrt{\frac{x(1+x)}{n}} + \frac{x(1+x)}{2n\delta}]\omega_1(f';\delta) \tag{3.12}$$

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