KYUNGPOOK Math. J. 35(1995), 17-23

Spectrum of positive definite functions on hypergroups

A. S. Okb El-Bab and F. M. Bayumi
Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City, Cairo, Egypt

(1991 AMS Classification number : 43A65)

Let K be a locally compact hypergroup as defined by R. Jewett. The purpose of this paper is to show that the amenability of K is equivalent to the folloiwng condition: "If φ is a continuous positive definite function defined on K and $\varphi \geq 0$ then the constant function 1_K belongs to the spectrum of φ". Our study deals with the cases of exponentially bounded hypergroups and discrete solvable hypergroups.

1. Introduction

There has recently been considerable interest shown by some harmonic analysts in the question of which topological spaces have enough structure so that a convolution on the corresponding space of all finite regular Borel measures can be defined. Dunkl [3], Jewett[5] and Spector [7] have all considered this question and they have given axioms which are essentially the same. These objects were called Hypergroups. We note that some methods of proof used in the group case are not available for hypergroups.

Let K be a locally compact Hausdorff space, $M(K)$ denote the space of all bounded randon measures, $M^1(K)$ be the subset of all probability measures and P_x be the point measure of $x \in K$. The support of a measure μ is denoted by *supp* μ. $C(K)$ denotes the space of continuous functions on K. The space K is called a hypergroup if the following conditions are satisfied:

(H1) There exists a map : $K \times K \to M^1(K)$, $(x, y) \to P_x * P_y$, called convolution, which is continuous, where $M^1(K)$ bears the vague topology.

(Received : 11 October 1993)

The linear extension to $M(K)$, see [5, Lemma 2.4B], satisfies

$$P_x * (P_y * P_z) = (P_x * P_y) * P_z$$

(H2) $suppP_x * P_y$ is compact.

(H3) There exists a homeomorphism $K \to K$, $x \to \bar{x}$, called involution, such that $x = \bar{\bar{x}}$ and $(P_x * P_y)^- = P_{\bar{y}} * P_{\bar{x}}$.

(H4) There exists an element $e \in K$, called unit element, such that $P_e * P_x = P_x * P_e = P_x$.

(H5) $e \in suppP_x * P_{\bar{y}}$ if and only if $x = y$.

(H6) The map $(x, y) \to suppP_x * P_y$ of $K \times K$ into the space of nonvoid compact subsets of K is continuous, the latter space with the topology as given in [5,7].

Now, let K be a hypergroup and $P(K)$ be the convex set of all continuous positive-definite functions ϕ on K with $\varphi(e) = 1$. The spectrum sp φ of $\varphi \in P(K)$ can be defined as the set of all indecomposable $\psi \in P(K)$ which are limits, in the sense of the topology of uniform convergence on compact subsets of K, of functions of the form

$$x \to \sum_{i,j=1}^{n} P_{x_i} * P_{\bar{x}_j}(\psi(x))c_i\bar{c}_j,$$

where $c_1, \cdots, c_n \in C$, $x_1, \cdots, x_n \in K$.

If π_φ denotes the cyclic unitary representation of K associated with φ, then $sp\ \varphi$ consists of all $\psi \in P(K)$ for which π_ψ is irreducible and weakly contained in π_φ (see [2], Chapter 18). If f is a positive function on K with compact support, $\tilde{f}(x) = \overline{f(x^{-1})}$ and $*$ is the convolution, it is easy to see that the function $f * \tilde{f}$ is positive-definite.

Our main subject here is to prove that exponentially bounded hypergroups and solvable discrete hypergroups satisfy the following property (which we denote by (P)):

(P) If $\varphi \in P(K)$ and if φ is positive in usual sense, then the constant positive-definite function 1 on K, 1_K, belongs to $sp\ \phi$. For connected hypergroups we show that the condition that the hypergroup is amenable is equivalent to the following weaker version (P^*) of P:

(P^*) If $\varphi \in P(K)$ and if φ is positive then $1_K \in sp_d(\varphi)$, where $sp_d(\varphi)$ is the spectrum of φ when the domain of φ is K_d (the discrete hypergroup).

2. Exponentially bounded hypergroups

Let π be a continuous unitary representation of the hypergroup K in the Hilbert space $< H_\pi, < \cdot, \cdot >)$. A unit vector $\xi \in H_\pi$ will be called a positive vector for π, if $Re\langle\pi(x)\xi, \xi\rangle \geq 0$ for all $x \in K$. So,

$$Re\langle\pi(\cdot)\xi, \xi\rangle \in P(K)$$

Now, it is easy to translate (P) into a property of unitary representations with positive vectors. In fact, consider the following property (P') of K which is formally stronger than (P):

(P') If π is a unitary representation of K with a positive vector, then π contains weakly 1_K.

Proposition 2.1. *(P) and (P') are equivalent for every hypergroup K.*

Proof. Let π be a unitary representation of K with a positive vector $\xi \in H_\pi$. Let $\varphi(x) = Re\langle\pi(x)\xi, \xi\rangle$, $x \in K$. If (P) holds, then 1_K is weakly contained in π_φ which is the subrepresentation of $\pi \oplus \pi$. Thus 1_K is weakly contained in $\pi \oplus \pi$ and this implies that 1_K is weakly contained in π.

A locally compact hypergroup is called Exponentially bounded if

$$\lim_n |G^n|^{1/n} = 1$$

for each compact neighbourhood G of e, where $|\cdot|$ denotes the Haar measure and $G^n = \{g_1 \cdots g_n; g_i \in G\}$. Exponentially bounded hypergroups are amenable [4].

Theorem 2.2. *Exponentially bounded hypergroups satisfy property (P).*

Proof. Let K be an exponentially bounded hypergroup and let $\varphi \in P(K)$ with $\varphi \geq 0$. Let G be a compact neighborhood of e with $G = G^{-1}$, and $\varepsilon > 0$. Then there is an $n \in N$ such that

$$\int_{G^{n+1} \times G^{n+1}} P_y * P_{\bar{z}}(\varphi) dy dz \leq (1 + \varepsilon) \int_{G^n \times G^n} P_y * P_{\bar{z}}(\varphi) dy dz, \qquad (1)$$

where dy and dz are Haar measures on K.

In fact, otherwise

$$|G^{n+1}|^2 \geq \int_{(G^{n+1})^2} P_y * P_{\bar{z}}(\varphi)dydz > (1+\varepsilon)^n \int_{G\times G} P_y * P_{\bar{z}}(\varphi)dydz$$

for all $n \in N$. Since

$$\int_{G\times G} P_y * P_{\bar{z}}(\varphi)dydz > 0,$$

this would be a contradiction with

$$\lim_n |G^n|^{1/n} = 1.$$

Now choose $n \in N$ such that (1) holds. Let $f = \chi_{G^n}$ be the characteristic function of G^n. Let π be the unitary representation of K associated to φ with Hilbert space H_π. Let $\xi \in H_\pi$ be such that $\varphi(x) = \langle \pi(x)\xi, \xi \rangle, x \in K$. Then

$$\|\pi(f)\xi\|^2 = \int_K f^* * f(x)\varphi(x)dx > 0,$$

since $f^* * f(e)\varphi(e) > 0$ and $f^* * f(x)\varphi(x) \geq 0$ for all $x \in K$.

Now let

$$\psi(x) = \frac{1}{\|\pi(f)\xi\|^2}\langle \pi(x)\pi(f)\xi, \pi(f)\xi \rangle, \quad x \in K.$$

Then ψ is associated to π. Moreover, for each $x \in K$

$$\begin{aligned} |\psi(x) - 1|^2 &= \frac{1}{\|\pi(f)\xi\|^4}|\langle \pi(xf - f)\xi, \pi(f)\xi \rangle|^2 \\ &\leq \frac{\|\pi(xf - f)\xi\|^2}{\|\pi(f)\xi\|^2} \\ &= \int_{K\times K}(xf - f)(y)(xf - f)(z)P_y * P_{\bar{z}}(\varphi)dydz / \\ &\quad \int_{K\times K} f(y)f(z)P_y * P_{\bar{z}}(\varphi)dydz \\ &= \int_{(xG^n \Delta G^n)^2} P_y * P_{\bar{z}}(\varphi)dydz \Big/ \int_{(G^n)^2} P_y * P_{\bar{z}}(\varphi)dydz, \end{aligned}$$

where $\triangle$ is the summmetric difference. Now (1) implies that for $x \in G$.

$$\begin{aligned}
&\int_{(xG^n \triangle G^n)^2} P_y * P_{\bar{z}}(\varphi) dydz \\
\leq\ & \int_{(G^{n+1})^2 \setminus (G^n)^2} P_y * P_{\bar{z}}(\varphi) dydy + \int_{(G^n \setminus xG^n)^2} P_y * P_{\bar{z}}(\varphi) dydy \\
\leq\ & \varepsilon \int_{(G^n)^2} P_y * P_{\bar{z}}(\varphi) dydz + \int_{(x^{-1}G^n \setminus xG^n)^2} P_y * P_{\bar{z}}(\varphi) dydz \\
\leq\ & 2\varepsilon \int_{(G^n)^2} P_y * P_{\bar{z}}(\varphi) dydz,
\end{aligned}$$

since $x^{-1} \in G$. Hence $|\psi(x) - 1|^2 \leq 2\varepsilon$ for all $x \in G$.

It is to be noted that Theorem 2.2 can be reformulate in the form : "If φ is positive and $\varphi \in P(K)$ where K is an exponentially bounded hypergroup, then the constant function 1_K is the uniform limit on compact subsets of K of functions of the form

$$x \to \sum_{i,j=1}^{n} P_{x_i} * P_{\bar{x}_j}(\varphi(x)) c_i \bar{c}_j,$$

where $c_l \geq 0$ and $x_l \in K$ for all $1 \leq l \leq n$.

Theorem 2.3. *Discrete solvable hypergroups satisfy property (P).*

Proof. Let K be a discrete solvable hypergroup and let $\varphi \in P(K)$ with $\varphi \geq 0$. Let $K = K_n \supseteq K_{n-1} \supseteq \cdots \supseteq K_0 = \{e\}$, be a composition series with abelian factor K_i/K_{i-1}, $1 \leq i \leq n$. First we show by induction on i that: for each $0 \leq i \leq n$ there is a net $(\psi_\alpha)_\alpha$ in $P(K)$ with $\psi \geq 0$ such that $\lim \psi_\alpha(x) = 1$ for all $x \in K_i$ and such that π_{ψ_α} is weakly contained in π for all α. For $i = 0$, the assertion is trivial (take $\psi_\alpha = \varphi$). For any i suppose that a net $(\psi_\alpha)_{\alpha \in N}$ exists. Let ψ be a limit point of $\{\psi_\alpha\}_{\alpha \in N}$ in the weak $*$-topology $\sigma(l^\infty(K), l^1(K))$. Then $\psi \in P(K)$ and $\psi \geq 0$. Moreover

$$\psi(x) = \lim_\alpha \psi_\alpha(x) = 1 \quad \text{for all } x \in K_i.$$

Hence $\psi | K_{i-1}$ factors to a positive definite function of K_{i+1}/K_i. Thus by Theorem 2.2 in its reformulated form there is a net $(\psi'_\beta)_\beta$ in $P(K_{i+1}/K_i)$

of the form

$$\psi'_\beta(x) = \sum c_k c_l P_{x_k} * P_{\bar{x}_l}(\psi(x)), \quad x \in K_{i+1},$$

where all $c_k \geq 0$ and $x_k \in K_{i+1}$, such that

$$\lim \psi'_\beta(x) = 1 \quad \text{for all } x \in K_{i+1}.$$

It is clear that $\psi'_\beta \in P(K)$ and $\psi'_\beta \geq 0$. Moreover $\pi_{\psi'_\beta} = \pi_\psi$. Hence each $\pi_{\psi'_\beta}$ is weakly contained in $\{\pi_{\psi_\alpha} | \alpha \in A\}$ which is weakly contained in π_φ. So, we get a net $(\psi_\alpha)_\alpha \in P(K)$ such that $\lim \psi_\alpha(x) = 1$ for all $x \in K_n = K$ and such that each π_{ψ_α} is weakly contained in π_φ. Hence 1_K is weakly contained in π_φ.

Now we reformulate property (P^*), defined earlier, as follows: If π is a unitary representation of K with positive vectors, then 1_K is weakly contained in π, when π and 1_K is viewed as representations of the discrete hypergroup K.

Theorem 2.4. *For a connected hypergroup K the following statements are equivalent:*

i) *K has property (P^*).*

ii) *K is amenable*

Proof. Suppose K is amenable. Let N be the closure of the commutative subhypergroup of K, by [1] proposition 3, N has polynomial growth hence it is exponentially bounded [4]. Let $\varphi \in P(K)$, $\varphi \geq 0$. By Theorem 2.2 in its reformulated form there is a net $(\psi_\alpha)_\alpha$ in $P(K)$ with $\psi_\alpha \geq 0$ such that $\lim \psi_\alpha(x) = 1$ for all $x \in N$ and such that π_{ψ_α} is weakly contained in π_φ for all α. Considering K as a discrete group we can apply the method of proof of theorem (2.4) to get some $\psi \in P(K)$, $\psi \geq 0$ with $\psi | N = 1$ and such that π_ψ is weakly contained in π_φ. Since K/N is abelian, 1_K is weakly contained in π_ψ and the result follows.

Now if K has property (P^*), then 1_K is weakly contained in the regular representation λ_k, when both representations are considered as representations of K. This is equivalent to the amenability of K [4].

References

[1] J. Boidol, Group Algebras with a unique C^*-Norm, J. Funct. Anal., 56(1984), 220-232.

[2] J. Dixmier, C^* Algebras, North-Holland, 1977.

[3] C. F. Dunkl, The Measure Algebra of a Locally Compact Hypergroup, Trans. Amer. Math. Soc., 179(1973), 331-348.

[4] Y. Guivarc'Il, Croissance polynomiale et pèriodes des Fonctions Harmoniques, Bull. Soc. Math. France, 101(1973), 333-379.

[5] R. I. Jewett, Spaces with an Abstract Convolution of Measures, Advances in Math., 18(1975), 1-101.

[6] J. P. Pier, Amenable Locally Compact Groups, Oxford University Press, 1984.

[7] R. Spector, Apercu de la Theorie des Hypergroups, Lecture Notes in Math., Vol. 497, Springer (1975), 643-673.