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# On Salagean- Pascu Type of Generalised Sakaguchi Class of Functions

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Let $A$ denote the class of functions $f$ analytic in the open unit disc $E = \{z \in C| \, |z| < 1\}$ normalized by $f(0) = f'(0) - 1 = 0$. Then $f \in A$ has the expansion $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. We define $I^n f(z) = z + \sum_{k=2}^{\infty} k^{-n} a_k z^k$ for all integer values of n. We observe that $I^{-n} f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k = D^n f(z)$ where $D$ is an operator defined by Salagoan. In this paper we define some new classes of functions using the differential operator $I^n$ and examine their properties.

## Introduction

Let $A$ be the class of functions $f$ satisfying $f(0) = f'(0) - 1 = 0$ and analytic in the unit disc $E = \{z \in C| \, |z| < 1\}$. Then $f \in A$ has the expansion $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. We define $I^n f(z) = z + \sum_{k=2}^{\infty} k^{-n} a_k z^k$ for all integer values of n. We then observe that $I^{-n} f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k = D^n f(z)$ where $D$ is an operator already defined in [7]. Also $I^{-1} f(z) = z f'(z) = D f(z)$ and $I^m (I^n f(z)) = I^{m+n} f(z)$.

**Definition.** Let $f$ and $g$ be two functions analytic in $E$ with $f(z) = \sum_{j=0}^{\infty} a_j z^j$ and $g(z) = \sum_{j=0}^{\infty} b_j z^j$. Then the Hadamard product or the convolution of $f$ and $g$ is given by $(f * g)(z) = \sum_{j=0}^{\infty} a_j b_j z^j$. The differential operator $I^n$ can also be seen as a convolution of two functions. Let

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$f \in A$. Then $I^n f(z) = \overbrace{k * k * \cdots k}^{n \text{ times}} * f(z)$ where $k(z) = z + \sum_{k=2}^{\infty} k^{-1} z^k = \log\left(\frac{1}{1-z}\right)$.

In this paper we define some new classes of functions using the differential operator $I^n$ and study their properties.

We state below two lemmas which are frequently applied in the sequel.

**Lemma A [1].** *Let $\beta, \gamma \in C$ and $h$ be a convex univalent function in $E$ with $h(0) = 1$ and $\mathrm{Re}(\beta h(z) + \gamma) > 0, z \in E$. Let $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$. Then $p(z) + \frac{zp'(z)}{(\beta p(z) + \gamma)} \prec h(z)$ implies $p(z) \prec h(z)$.*

**Lemma B [2].** *Let $\beta, \gamma \in C$. Let $h$ be a convex univalent function in $E$ with $h(0) = 1$ and $\mathrm{Re}(\beta h(z) + \gamma) > 0, z \in E$, and let $q$ be an analytic function in $E$ with $q(0) = 1$ and $q(z) \prec h(z), z \in E$. If $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ is analytic in $E$ then $p(z) + \frac{zp'(z)}{(\beta q(z) + \gamma)} \prec h(z)$ implies $p(z) \prec h(z)$.*

## Main Results

**Definition 1.** Any $f \in A$ with $f(z)f'(z)/z \neq 0$ in $E$ is said to be in $S_s^n(\alpha)$ if it satisfies

$$\mathrm{Re}\left(\frac{\alpha I^{n-2} f(z) + (1-\alpha) I^{n-1} f(z)}{\alpha I^{n-1} g(z) + (1-\alpha) I^n g(z)}\right) > 0 \text{ in } E$$

where $g(z) = (f(z) - f(-z))/2$.

When $\alpha = 1, n = 1$ we get a class defined by Sakaguchi [6], and when $n = 1$ we obtain a class defined by Radha [3].

**Theorem 1.** *Let $f \in S_S^n(\alpha)$. Then*

$$\mathrm{Re}\left(\frac{\alpha I^{n-2} g(z) + (1-\alpha) I^{n-1} g(z)}{\alpha I^{n-1} g(z) + (1-\alpha) I^n g(z)}\right) > 0$$

*Further if $0 \leq \alpha \leq 1, \mathrm{Re}(I^{n-1} g(z) / I^n g(z)) > 0$.*

*Proof.* Consider

$$\begin{aligned}
&\frac{\alpha I^{n-2}g(z)+(1-\alpha)I^{n-1}g(z)}{\alpha I^{n-1}g(z)+(1-\alpha)I^{n}g(z)}\\
&= \frac{\alpha I^{n-2}(f(z)-f(-z))+(1-\alpha)I^{n-1}(f(z)-f(-z))}{2(\alpha I^{n-1}g(z)+(1-\alpha)I^{n}g(z))}\\
&= \frac{\alpha I^{n-2}f(z)+(1-\alpha)I^{n-1}f(z)}{2(\alpha I^{n-1}g(z)+(1-\alpha)I^{n}g(z))}-\frac{\alpha I^{n-2}f(-z)+(1-\alpha)I^{n-1}f(-z)}{2(\alpha I^{n-1}g(z)+(1-\alpha)I^{n}g(z))}
\end{aligned}$$

Now

$$\begin{aligned}
g(z) &= \frac{1}{2}\left[z+\sum_{k=2}^{\infty}a_k z^k-\left((-z)+\sum_{k=2}^{\infty}a_k(-z)^k\right)\right]\\
&= z+\sum_{k=2}^{\infty}a_{2k-1}z^{2k-1} \text{ is an odd function of z.}
\end{aligned}$$

Hence we see that

$$\begin{aligned}
&\mathrm{Re}\left(\frac{\alpha I^{n-2}g(z)+(1-\alpha)I^{n-1}g(z)}{\alpha I^{n-1}g(z)+(1-\alpha)I^{n}g(z)}\right)\\
&= \mathrm{Re}\left(\frac{\alpha I^{n-2}f(z)+(1-\alpha)I^{n-1}f(z)}{2(\alpha I^{n-1}g(z)+(1-\alpha)I^{n}g(z))}+\frac{\alpha I^{n-2}f(-z)+(1-\alpha)I^{n-1}f(-z)}{2(\alpha I^{n-1}g(z)+(1-\alpha)I^{n}g(z))}\right)\\
&= \mathrm{Re}\left(\frac{\phi(z)+\phi(-z)}{2}\right)>0\\
&\quad \text{where } \phi(z)=\frac{\alpha I^{n-2}f(z)+(1-\alpha)I^{n-1}f(z)}{2(\alpha I^{n-1}g(z)+(1-\alpha)I^{n}g(z))}
\end{aligned}$$

Since $f\in S_s^n(\alpha)$.

Let $R=\sup\{r|I^n g(z)\neq 0 \text{ in } 0<|z|<r\}$. We set

$$I^{n-1}g(z)/I^n g(z)=p(z);$$

hence

$$I^{n-1}g(z)=p(z)I^n g(z).$$

Differentiating this with respect to $z$ and then multiplying the equation by $z$ we get

$$z(I^{n-1}g(z))' = z(I^n g(z)p(z))'.$$

Since $I^{-1}f(z) = zf'(z)$,

$$I^{-1}(I^{n-1}g(z)) = zp'(z)I^n g(z) + p(z)I^{-1}(I^n g(z));$$

or

$$I^{n-2}g(z) = zp'(z)I^n g(z) + p(z)I^{n-1}g(z).$$

Thus

$$\begin{aligned} &\operatorname{Re}\left\{\frac{\alpha I^{n-2}g(z) + (1-\alpha)I^{n-1}g(z)}{\alpha I^{n-1}g(z) + (1-\alpha)I^n g(z)}\right\} \\ = \;& \operatorname{Re}\left\{\frac{\alpha z p'(z)I^n g(z) + \alpha p(z)I^{n-1}g(z) + (1-\alpha)I^{n-1}g(z)}{\alpha I^{n-1}g(z) + (1-\alpha)I^n g(z)}\right\} \\ = \;& \operatorname{Re}\left\{p(z) + \frac{zp'(z)}{p(z) + \frac{1-\alpha}{\alpha}}\right\} > 0 \end{aligned}$$

by the first part of the theorem.

Applying Lemma A with $h(z) = (1+z)/(1-z)$, we have $\operatorname{Re} p(z) > 0$ provided $\alpha \le 1$.

That is whenever $\alpha \le 1$ and $f \in S_s^n(\alpha)$, $\operatorname{Re}\left(I^{n-1}g(z)/I^n g(z)\right) > 0$ in $|z| < r$. Hence $I^n g(z)$ is starlike in $|z| < r$ or $I^n g(z)$ is univalent in $|z| < r$ and cannot vanish in $|z| = r < 1$.

Thus we conclude that $R = 1$ and the proof is complete.

**Theorem 2.** *Let $f \in S_s^{n-1}(\alpha)$ then $f \in S_s^n(\alpha)$.*

*Proof.* $f \in S_s^{n-1}(\alpha)$ implies that $f$ satisfies

$$\operatorname{Re}\left\{\frac{\alpha I^{n-3}f(z) + (1-\alpha)I^{n-2}f(z)}{\alpha I^{n-2}g(z) + (1-\alpha)I^{n-1}g(z)}\right\} > 0$$

where $g(z)$ is defined as before. Let

$$p(z) = \frac{\alpha I^{n-2}f(z) + (1-\alpha)I^{n-1}f(z)}{\alpha I^{n-1}g(z) + (1-\alpha)I^n g(z)}$$

and

$$q(z)=\frac{\alpha I^{n-2}g(z)+(1-\alpha)I^{n-1}g(z)}{\alpha I^{n-1}g(z)+(1-\alpha)I^{n}g(z)}.$$

Then

$$q(z)(\alpha I^{n-1}g(z)+(1-\alpha)I^{n}g(z))=\alpha I^{n-2}g(z)+(1-\alpha)I^{n-1}g(z).$$

Differentiating this with respect to $z$ and multiplying by $z$ (noting that $zf'(z)=I^{-1}f(z)$) we get

$$zq'(z)(\alpha I^{n-1}g(z)+(1-\alpha)I^{n}g(z))+q(z)(\alpha I^{n-2}g(z)+(1-\alpha)I^{n-1}g(z))$$
$$=\alpha I^{n-3}g(z)+(1-\alpha)I^{n-2}g(z);$$

or

$$\mathrm{Re}\left(\frac{zq'(z)}{q(z)}+q(z)\right)=\mathrm{Re}\left(\frac{\alpha I^{n-3}g(z)+(1-\alpha)I^{n-2}g(z)}{\alpha I^{n-2}g(z)+(1-\alpha)I^{n-1}g(z)}\right)>0$$

by Theorem 1, for $f\in S_s^{n-1}(\alpha)$. Now an application of Lemma A with $\beta=1,\gamma=0$ and $h(z)=\frac{1+z}{1-z}$ gives $\mathrm{Re}q(z)>0$ in $E$.

Also

$$p(z)[\alpha I^{n-1}g(z)+(1-\alpha)I^{n}g(z)]=\alpha I^{n-2}f(z)+(1-\alpha)I^{n-1}f(z).$$

Differentiating this with respect to $z$ and noting that $zf'(z)=I^{-1}f(z)$ we have

$$\begin{aligned}&zp'(z)[\alpha I^{n-1}g(z)+(1-\alpha)I^{n}g(z)]\\&\quad+p(z)I^{-1}[\alpha I^{n-1}g(z)+(1-\alpha)I^{n}g(z)]\\=\;&I^{-1}[\alpha I^{n-2}f(z)+(1-\alpha)I^{n-1}f(z)];\end{aligned}$$

or

$$\begin{aligned}&zp'(z)[\alpha I^{n-1}g(z)+(1-\alpha)I^{n}g(z)]\\&\quad+p(z)[\alpha I^{n-2}g(z)+(1-\alpha)I^{n-1}g(z)]\\=\;&\alpha I^{n-3}f(z)+(1-\alpha)I^{n-2}f(z)].\end{aligned}$$

This yields

$$\frac{\alpha I^{n-3}f(z)+(1-\alpha)I^{n-2}f(z)}{\alpha I^{n-2}g(z)+(1-\alpha)I^{n-1}g(z)}=p(z)+\frac{zp'(z)}{q(z)}.$$

Since $f\in S_s^{n-1}(\alpha), \mathrm{Re}\,(p(z)+zp'(z)/q(z))>0$ and an application of Lemma B gives $\mathrm{Re}p(z)>0$ and the theorem is thus proved.

**Theorem 3**. *The inclusion relation $S_s^n(\alpha)\subset S_s^n(0)$ is satisfied for $0\le\alpha\le 1$.*

*Proof.* Let $f\in S_s^n(\alpha)$. We set

$$\frac{I^{n-1}f(z)}{I^n g(z)}=p(z)\ \text{ and }\ \frac{I^{n-1}g(z)}{I^n g(z)}=q(z),$$

where $g(z)=(f(z)-f(-z))/2$.

From theorem 1 we infer that Re $q(z)>0$. Now

$$\begin{aligned}&\frac{\alpha I^{n-2}f(z)+(1-\alpha)I^{n-1}f(z)}{\alpha I^{n-1}g(z)+(1-\alpha)I^n g(z)}\\ =&\ \frac{I^{-1}[\alpha pI^n g(z)]+(1-\alpha)p(z)K^n g(z)}{\alpha I^{n-1}g(z)+(1-\alpha)I^n g(z)}\\ =&\ \frac{\alpha p(z)I^{n-1}g(z)+\alpha zp'(z)I^n g(z)+(1-\alpha)p(z)I^n g(z)}{\alpha I^{n-1}g(z)+(1-\alpha)I^n g(z)}\\ =&\ p(z)+\frac{zp'(z)}{q(z)+\frac{(1-\alpha)}{\alpha}}.\end{aligned}$$

Since $f\in S_s^n(\alpha)$,

$$\mathrm{Re}\ p(z)+\frac{zp'(z)}{q(z)+\frac{(1-\alpha)}{\alpha}}>0.$$

An application of Lemma B gives that Re $p(z)>0$ if $\alpha\le 1$ thereby proving the theorem.

**Theorem 4**. *Let $f\in S_s^n(\alpha)$. If $F$ is defined by the equation*

$$F(z)=\frac{1}{\alpha z^{1/\alpha-1}}\int_0^z t^{\frac{1}{\alpha}-2}f(t)dt$$

*then* $F \in S_s^n(\alpha)$ *for* $0 < \alpha \leq 1$.

*Proof.* From the definition of $F(z)$ we get

$$z^{\frac{1}{\alpha}-1}F(z) = \frac{1}{\alpha}\int_0^z t^{\frac{1}{\alpha}-2}f(t)dt.$$

Differentiating with respect to $z$,

$$z^{\frac{1}{\alpha}-1}F'(z) + (\frac{1}{\alpha} - 1)z^{\frac{1}{\alpha}-2}F(z) = \frac{1}{\alpha}z^{\frac{1}{\alpha}-2}f(z).$$

That is

$$\alpha zF'(z) + (1-\alpha)F(z) = f(z). \tag{1}$$

Now

$$\begin{aligned} g(z) &= \frac{f(z) - f(-z)}{2} \\ &= \frac{1}{2}[\alpha zF'(z) + (1-\alpha)F(z) - \alpha(-z)F'(-z) - (1-\alpha)F(-z)] \\ &= \frac{\alpha}{2}(zF'(z) - (-z)F(-z)) + \frac{(1-\alpha)}{2}(F(z) - F(-z)) \\ &= \frac{1}{2}(\alpha zG'(z) + (1-\alpha)G(z)), \end{aligned} \tag{2}$$

where $G(z) = \frac{F(z)-F(-z)}{2}$.

Also we can see that

$$G(z) = \frac{1}{\alpha z^{1/\alpha-1}}\int_0^z t^{\frac{1}{\alpha}-2}g(t)dt.$$

By Theorem 1, $g(z) \neq 0$ in $E - \{0\}$; hence equation (2) implies that $[\alpha zG'(z) + (1-\alpha)G(z)]/2$ does not vanish in $E - \{0\}$. Using equation (1) and (2) we get

$$\frac{\alpha I^{n-2}F(z) + (1-\alpha)I^{n-1}F(z)}{\alpha I^{n-1}G(z) + (1-\alpha)I^nG(z)} = \frac{I^{n-1}f(z)}{I^ng(z)}.$$

Thus whenever $f \in S_s^n(\alpha)$, from Theorem 3 we have

$$\text{Re}\{I^{n-1}f(z)/I^ng(z)\}0.$$

Hence $F \in S_s^n(\alpha)$ for $0 < \alpha \leq 1$.

**Definition 2.** Any $f \in A$ with $f'(z)f(z)/z \neq 0$ in $E$ is said to belong to the class $C_s^n(\alpha)$ if it satisfies

$$\mathrm{Re}\left\{\frac{\alpha I^{n-2}f(z) + (1-\alpha)I^{n-1}f(z)}{\alpha I^{n-2}\psi(z) + (1-\alpha)I^n\psi(z)}\right\} > 0, \quad z \in E,$$

where $\psi(z) = (\phi(z) - \phi(-z))/2$ for some $\phi \in S_s^n(\alpha)$.

**Theorem 5.** *Let $f \in C_s^{n-1}(\alpha)$. Then $f \in C_s^n(\alpha)$.*

*Proof.* $f \in C_s^{n-1}(\alpha)$ implies that there exists a $\phi \in S_s^n(\alpha)$ such that

$$\mathrm{Re}\left\{\frac{\alpha I^{n-3}f(z) + (1-\alpha)I^{n-2}f(z)}{\alpha I^{n-2}\psi(z) + (1-\alpha)I^{n-1}\psi(z)}\right\} > 0$$

where $\psi(z) = (\phi(z) - \phi(-z))/2$. Let

$$p(z) = \frac{\alpha I^{n-2}f(z) + (1-\alpha)I^{n-1}f(z)}{\alpha I^{n-1}\psi(z) + (1-\alpha)I^n\psi(z)}$$

and

$$q(z) = \frac{\alpha I^{n-2}\psi(z) + (1-\alpha)I^{n-1}\psi(z)}{\alpha I^{n-1}\psi(z) + (1-\alpha)I^n\psi(z)}.$$

By Theorem 1, $\psi \in S_s^n(\alpha)$ and hence Re $q(z) > 0$. Further we have

$$p(z)[\alpha I^{n-1}\psi(z) + (1-\alpha)I^n\psi(z)] = \alpha I^{n-2}f(z) + (1-\alpha)I^{n-1}f(z).$$

Differentiating with respect to $z$ and using the relation $I^{-1}f(z) = zf'(z)$ we get,

$$zp'(z)[\alpha I^{n-1}\psi(z) + (1-\alpha)I^n\psi(z)] + p(z)[\alpha I^{n-2}\psi(z) + (1-\alpha)I^{n-1}\psi(z)]$$
$$= \alpha I^{n-3}f(z) + (1-\alpha)I^{n-2}f(z)$$

Hence

$$p(z) + \frac{zp'(z)}{q(z)} = \frac{\alpha I^{n-3}f(z) + (1-\alpha)I^{n-2}f(z)}{\alpha I^{n-2}\psi(z) + (1-\alpha)I^{n-1}\psi(z)}.$$

Since $f \in C_s^{n-1}(\alpha)$, Re $\{p(z) + \frac{zp'(z)}{q(z)}\} > 0$ and now an application of Lemma B gives Re $p(z) > 0$ and this proves the theorem.

We now proceed to define two generalised classes of functins with respect to conjugate points and study their properties.

**Definition 3.** Any $f \in A$ with $f'(z)f(z)/z \neq 0$ in $E$ is said to be in $S_c^n(\alpha)$ if it satisfies

$$\text{Re}\left\{\frac{\alpha I^{n-2}f(z)+(1-\alpha)I^{n-1}f(z)}{\alpha I^{n-1}g(z)+(1-\alpha)I^{n}g(z)}\right\}>0$$

where $g(z)=(f(z)+\overline{f(\bar{z})})/2$.

When $\alpha = 0, n = 0$ we get a class defined in [4]; when $n = 0$ this class reduces to a class defined by S. Radha [5].

**Theorem 6.** *Let $f \in S_c^n(\alpha)$. Then*

$$\text{Re}\left\{\frac{\alpha I^{n-2}g(z)+(1-\alpha)I^{n-1}g(z)}{\alpha I^{n-1}g(z)+(1-\alpha)I^{n}g(z)}\right\}>0$$

*where $g$ is defined as above. Further if $0 \leq \alpha \leq 1$, $Re\{I^{n-1}g(z)/I^{n}g(z)\} > 0$.*

*Proof.* Consider

$$\begin{aligned}
&\frac{\alpha I^{n-2}g(z)+(1-\alpha)I^{n-1}g(z)}{\alpha I^{n-1}g(z)+(1-\alpha)I^{n}g(z)}\\
=\;&\frac{\alpha I^{n-2}(f(z)+\overline{f(\bar{z})})+(1-\alpha)I^{n-1}(f(z)+\overline{f(\bar{z})})}{2(\alpha I^{n-1}g(z)+(1-\alpha)I^{n}g(z))}\\
=\;&\frac{\alpha I^{n-2}f(z)+(1-\alpha)I^{n-1}f(z)}{2(\alpha I^{n-1}g(z)+(1-\alpha)I^{n}g(z))}\\
&+\frac{\alpha I^{n-2}\overline{f(\bar{z})}+(1-\alpha)I^{n-1}\overline{f(\bar{z})}}{2(\alpha I^{n-1}g(z)+(1-\alpha)I^{n}g(z))}.
\end{aligned}\tag{3}$$

We have

$$\begin{aligned}
\alpha I^{n-2}\overline{f(\bar{z})}+(1-\alpha)I^{n-1}\overline{f(\bar{z})}) &= \sum_{k=0}^{\infty}[\alpha k^{-(n-2)}+(1-\alpha)k^{-(n-1)}]\bar{a}_k z_k\\
&= \overline{(\alpha I^{n-2}+(1-\alpha)I^{n-1})f(\bar{z})}
\end{aligned}$$

since $\alpha$ and $k$ are reals.

Further, since $\overline{g(\bar{z})} = g(z)$ we get

$$\frac{\alpha I^{n-2}\overline{f(\bar{z})} + (1-\alpha)I^{n-1}\overline{f(\bar{z})}}{2[\alpha I^{n-1}g(z) + (1-\alpha)I^{n}g(z)]} = \overline{\frac{\alpha I^{n-2}f(\bar{z}) + (1-\alpha)I^{n-1}f(\bar{z})}{2[\alpha I^{n-1}g(\bar{z}) + (1-\alpha)I^{n}g(\bar{z})]}}.$$

Hence the right and side of equation (3) becomes

$$\begin{aligned} &\frac{\alpha I^{n-2}f(z) + (1-\alpha)I^{n-1}f(z)}{2[\alpha I^{n-1}g(z) + (1-\alpha)I^{n}g(z)]} + \overline{\frac{\alpha I^{n-2}f(\bar{z}) + (1-\alpha)I^{n-1}f(\bar{z})}{2[\alpha I^{n-1}g(\bar{z}) + (1-\alpha)I^{n}g(\bar{z})]}} \\ &= \frac{\phi(z) + \overline{\phi(\bar{z})}}{2} \end{aligned}$$

where $\phi(z) = \frac{\alpha I^{n-2}f(z) + (1-\alpha)I^{n-1}f(z)}{\alpha I^{n-1}g(z) + (1-\alpha)I^{n}g(z)}$.

Since $f \in S_s^n(\alpha)$, Re $\phi(z) > 0$ and also Re $\overline{\phi(\bar{z})} > 0$; this implies

$$\text{Re}\left\{\frac{\alpha I^{n-2}g(z) + (1-\alpha)I^{n-1}g(z)}{\alpha I^{n-1}g(z) + (1-\alpha)I^{n}g(z)}\right\} > 0.$$

Let $R = \{\sup r : I^n g(z) \neq 0 \text{ in } 0 < |z| < r\}$. We set

$$\frac{I^{n-1}g(z)}{I^n g(z)} = p(z)$$

and hence

$$I^{n-1}g(z)p(z)I^n g(z).$$

Differentiating this with respect to $z$ and nothing that $I^{-1}f(z) = zf'(z)$ we get

$$z(I^{n-1}g(z))' = zp'(z)I^n g(z) + p(z)z(I^n(g(z))',$$

that is

$$I^{n-2}g(z)) = zp'(z)I^n g(z) + p(z)I^{n-1}g(z).$$

Thus

$$\text{Re}\left\{\frac{\alpha I^{n-2}g(z) + (1-\alpha)I^{n-1}g(z)}{\alpha I^{n+1}g(z) + (1-\alpha)I^{n}g(z)}\right\}$$

$$= \quad \text{Re} \left\{ \frac{\alpha(zp'(z)I^n g(z) + p(z)I^{n-1}g(z)) + (1-\alpha)p(z)I^n g(z)}{\alpha I^{n-1}g(z) + (1-\alpha)I^n g(z)} \right\}$$

$$= \quad \text{Re} \left\{ p(z) + \frac{zp'(z)}{p(z) + \frac{(1-\alpha)}{\alpha}} \right\} > 0$$

by the first part of the theroem.

Now applying Lemma A with $h(z) = (1+z)/(1-z)$ we have Re $p(z) > 0$ provided $0 \leq \alpha \leq 1$.

Thus Re $\{I^{n-1}g(z)/I^n g(z)\} > 0$ in $|z| < R$. Hence $I^n g(z)$ is starlike in $|z| < R$ or $I^n g(z)$ is univalent in $|z| < R$ and cannot vanish on $|z| = R < 1$. We conclude that $R = 1$ and the proof is complete.

**Theorem 7.** *Let $f \in S_c^{n-1}(\alpha)$. Then $f \in S_c^n(\alpha)$.*

*Proof.* $f \in S_c^{n-1}(\alpha)$ implies that

$$\text{Re}\left\{ \frac{\alpha I^{n-3}f(z) + (1-\alpha)I^{n-2}f(z)}{\alpha I^{n-2}g(z) + (1-\alpha)I^{n-1}g(z)} \right\} > 0$$

where $g(z)$ is defined as in Definition 3. Let

$$p(z) = \frac{\alpha I^{n-2}f(z) + (1-\alpha)I^{n-1}f(z)}{\alpha I^{n-1}g(z) + (1-\alpha)I^n g(z)}$$

and

$$q(z) = \frac{\alpha I^{n-2}g(z) + (1-\alpha)I^{n-1}g(z)}{\alpha I^{n-1}g(z) + (1-\alpha)I^n g(z)}.$$

Using Teorem 6 for $f \in S_c^{n-1}(\alpha)$ and Lemma A, we can show that $Req(z) > 0$ as in Theorem 2.

Now

$$p(z)[2I^{n-1}g(z) + (1-\alpha)I^n g(z)] = 2I^{n-2}f(z) + (1-\alpha)I^{n-1}f(z).$$

Differentiating this with respect to $z$ and using the fact that $I^{-1}f(z) = zf'(z)$ we get

$$zp'(z)[\alpha I^{n-1}g(z) + (1-\alpha)I^n g(z)] + p(z)[\alpha I^{n-2}g(z) + (1-\alpha)I^{n-1}g(z)$$
$$= \alpha I^{n-3}f(z) + (1-\alpha)I^{n-2}f(z)$$

Hence

$$\operatorname{Re}\left\{\frac{\alpha I^{n-3}f(z)+(1-\alpha)I^{n-2}f(z)}{\alpha I^{n-2}g(z)+(1-\alpha)I^{n-1}g(z)}\right\}=\operatorname{Re}\ \{p(z)+\frac{zp'(z)}{q(z)}\}.$$

Since $f\in S_c^{n-1}(\alpha)$ we see that Re $\{p(z)+\frac{zp'(z)}{q(z)}\}>0$ and an application of Lemma B gives Re $p(z)>0$ and thus the theorem is proved.

**Theorem 8.** *The inclusion relation $S_c^n(\alpha)\subset S_c^n(0)$ is satisfied for $0\leq\alpha\leq 1$.*

*Proof.* Let $\frac{I^{n-1}f(z)}{I^n g(z)}=p(z)$ and $\frac{I^{n-1}g(z)}{I^n g(z)}=q(z)$ where $g(z)=\frac{f(z)+\overline{f(\bar{z})}}{2}$. From Theorem 6 we infer that Re $q(z)>0$. Now

$$\begin{aligned}
&\frac{\alpha I^{n-2}f(z)+(1-\alpha)I^{n-1}f(z)}{\alpha I^{n-1}g(z)+(1-\alpha)I^n g(z)}\\
=&\ \frac{\alpha I^{n-1}(p(z)I^n g(z))+(1-\alpha)p(z)I^n g(z)}{\alpha I^{n-1}g(z)+(1-\alpha)I^n g(z)}\\
=&\ \frac{\alpha p(z)I^{n-1}g(z)+\alpha zp'(z)I^n g(z)+(1-\alpha)p(z)I^n g(z)}{\alpha I^{n-1}g(z)+(1-\alpha)I^n g(z)}\\
=&\ p(z)+\frac{zp'(z)}{q(z)+\frac{(1-\alpha)}{\alpha}}
\end{aligned}$$

$f\in S_c^n(\alpha)$ implies that Re $\left\{p(z)+\frac{zp'(z)}{q(z)+\frac{1-\alpha}{\alpha}}\right\}>0$.

Now applying Lemma B we infer that Re $p(z)>0$ and the theorem follows.

**Theorem 9.** *Let $f\in S_c^n(\alpha)$. If $F$ is defined by the equation*

$$F(z)=\frac{1}{\alpha z^{1/\alpha-1}}\int_0^2 t^{\frac{1}{\alpha}-2}f(t)dt$$

*then $F\in S_c^n(\alpha)$.*

*Proof.* From the definition of $F(z)$ we get

$$z^{\frac{1}{\alpha}-1}F(z)=\frac{1}{\alpha}\int_0^z t^{\frac{1}{\alpha}-2}f(t)dt.$$

Differentiating with respect to $z$ we obtain

$$z^{\frac{1}{\alpha}-1}F'(z)+(\frac{1}{\alpha}-1)z^{\frac{1}{\alpha}-2}F(z)=\frac{1}{\alpha}z^{\frac{1}{\alpha}-2}F(z)$$

or

$$zF'(z)+(1-\alpha)F(z)=f(z). \tag{4}$$

Also

$$\begin{aligned} g(z) &= \frac{f(z)+\overline{f(\bar{z})}}{2} \\ &= \frac{1}{2}[\alpha zF'(z)+(1-\alpha)F(z)+\alpha z\overline{F'(\bar{z})}+(1-\alpha)\overline{F'(\bar{z})}] \\ &= \frac{\alpha}{2}[\alpha z(F'(z)+\overline{F'(\bar{z})})+(1-\alpha)(F(z)+\overline{F(\bar{z})})] \\ &= \alpha zG'(z)+(1-\alpha)G(z), \end{aligned} \tag{5}$$

where $G(z)=\frac{(F(z)+\overline{F(\bar{z})})}{2}$. Now $g(z)\neq 0$ in $E-\{0\}$ by Theorem 6) and hence $\alpha zG'(z)+(1-\alpha)G(z)$ does not vanish in $E-\{0\}$. Also it is clear that

$$G(z)=\frac{1}{\alpha z^{1/\alpha-1}}\int_0^z t^{\frac{1}{\alpha}-2}g(t)dt.$$

Using equation (4) and (5) we get

$$\frac{\alpha I^{n-2}F(z)+(1-\alpha)I^{n-1}F(z)}{\alpha I^{n-1}G(z)+(1-\alpha)I^{n}G(z)}=\frac{I^{n-1}f(z)}{I^{n}g(z)}.$$

Whenever $f\in S_c^n(\alpha)$, applying Theorem 8 we infer that $F\in S_c^n(\alpha)$ for $0<\alpha\leq 1$.

**Definition 4.** Any $f\in A$ with $f'(z)f(z)/z\neq 0$ in $E$ is said to belong to the class $C_c^n(\alpha)$ if it satisfies

$$\operatorname{Re}\left\{\frac{\alpha I^{n-2}f(z)+(1-\alpha)I^{n-1}f(z)}{\alpha I^{n-1}\psi(z)+(1-\alpha)I^{n-1}\psi(z)}\right\}>0, \quad z\in E,$$

where $\psi(z)=(\phi(z)+\overline{\phi(\bar{z})})/2$ for some $\phi\in S_c^n(\alpha)$.

**Theorem 10.** *Let $f\in C_c^{n-1}(\alpha)$. Then $f\in C_c^n(\alpha)$.*

*Proof.* $f \in C_c^{n-1}(\alpha)$ implies that there exists a $\phi \in S_c^n(\alpha)$ such that

$$\text{Re}\left\{\frac{\alpha I^{n-3}f(z)+(1-\alpha)I^{n-2}f(z)}{\alpha I^{n-2}\psi(z)+(1-\alpha)I^{n-1}\psi(z)}\right\} > 0$$

where $\psi(z) = (\phi(z)+\overline{\phi(\bar{z})})/2$. We set

$$p(z) = \frac{\alpha I^{n-2}f(z)+(1-\alpha)I^{n-1}f(z)}{\alpha I^{n-1}\psi(z)+(1-\alpha)I^{n}\psi(z)}$$

and

$$q(z) = \frac{\alpha I^{n-2}\psi(z)+(1-\alpha)I^{n-1}\psi(z)}{\alpha I^{n-1}\psi(z)+(1-\alpha)I^{n}\psi(z)}.$$

From the definition of the function $\psi$ and by Theorem 6, we have Re $q(z) > 0$. We have

$$p(z)[\alpha I^{n-1}\psi(z)+(1-\alpha)I^{n}\psi(z)] = \alpha I^{n-2}f(z)+(1-\alpha)I^{n-1}f(z).$$

Differentiating this with respect to $z$ and using $I^{-1}f(z) = zf'(z)$ we get

$$\begin{aligned} &zp'(z)[\alpha I^{n-1}\psi(z)+(1-\alpha)I^{n}\psi(z)] + p(z)[\alpha I^{n-2}\psi(z)+(1-\alpha)I^{n-1}\psi(z)] \\ &= \alpha I^{n-3}f(z)+(1-\alpha)I^{n-2}f(z). \end{aligned}$$

This gives

$$p(z)+\frac{zp'(z)}{q(z)} = \frac{\alpha I^{n-3}f(z)+(1-\alpha)I^{n-2}f(z)}{\alpha I^{n-2}\psi(z)+(1-\alpha)I^{n-1}\psi(z)}.$$

Since $f \in C_c^{n-1}(\alpha)$, Re $\{p(z)+\frac{zp'(z)}{q(z)}\} > 0$ and now an application of Lemma B gives Re $p(z) > 0$ there by proving the theorem.

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