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# Reliability Estimation for a Shared-Load System Based on Freund Model

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**Abstract** This paper considers the reliability estimation of a two-component shared-load system based on Freund model. Maximum likelihood estimator, order restricted maximum likelihood estimator and uniformly minimum variance unbiased estimator of the reliability function for the system are obtained. Performance of three estimators for moderate sample sizes is studied by simulation.

*Key words* : Shared-load model, Reliability function, UMVUE, Order restricted MLE

## 1. Introduction

The quantification of the reliability of parallel systems is based on the assumption that, when a redundant component fails, the failure rate or the reliability of the surviving components is not affected by the failed component. In some situations, however, all the components share the load during the mission and the failure rate of the surviving components may increase due to increased load when a component fails. Systems such as a multi-processor computer and electric generators sharing an electrical load in plant can be described by a shared-load model. To correctly determine the reliability of such systems, the increase of failure rate of the surviving components has to be considered.

Freund(1961) proposed a bivariate extension of the exponential distribution by allowing the failure rate of the surviving component to be changed after the failure of one component. The Freund model applies, in particular, to two-component shared-load system that can function even if one component has failed. He also obtained the maximum likelihood estimators(MLE's) of the model parameters. Weier(1981) obtained Bayes estimators of parameters and reliability function for the Freund model.

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In this paper we obtain MLE, order restricted maximum likelihood estimator (OMLE), and uniformly minimum variance unbiased estimator(UMVUE) of the reliability function for the Freund model. Finally, numerical studies are carried out to compare these estimators.

## 2. Reliabilty Estimation for a Shared-load Model

Consider a system which functions when at least one of the two identical components functions. Let $Z_1$ and $Z_2$ are independently and identically distributed random variables(rv's) with constant failure rate $\lambda(>0)$.

Let $X_1 = \min(Z_1, Z_2)$ and $X_2 = \max(Z_1, Z_2)$. Then $X_1$ and $(X_2 - X_1)$ are independent exponential rv's with failure rates $2\lambda$ and $\lambda_1$, respectively, where $\lambda_1(\geq \lambda)$ is the failure rate of a surviving component when one of the two components fails at $X_1$.

System failure occurs at time $X_2$. Then the reliability of a two-component shared-load system is given by (see Scheuer(1988) and Lin et al.(1993))

$$R(t) = \Pr(X_2 > t) \tag{1a}$$

$$= \begin{cases} \dfrac{\lambda_1}{\lambda_1 - 2\lambda} e^{-2\lambda t} + \dfrac{2\lambda}{2\lambda - \lambda_1} e^{-\lambda_1 t}, & if\ 2\lambda \neq \lambda_1 \\ (1 + 2\lambda_c t) e^{-2\lambda_c t}, & if\ 2\lambda = \lambda_1 \end{cases} \tag{1b}$$

where $\lambda_c = \lambda$ when $2\lambda = \lambda_1$.

Suppose $n$ systems each with 2 components are put on test. Let $Y_{1j} = X_{1j}$ and $Y_{2j} = X_{2j} - X_{1j}$, where $X_{1j} = \min(Z_{1j}, Z_{2j})$ and $X_{2j} = \max(Z_{1j}, Z_{2j})$ denote the ordered lifetimes observed on $j-th$ system, $j = 1, 2, \cdots, n$. Then $Y_{1j}$ and $Y_{2j}$ are independent exponential rv's with failure rates $2\lambda$ and $\lambda_1$, respectively. The likelihood function can be written as

$$L = \begin{cases} (2\lambda\lambda_1)^n \exp(-2\lambda t_1 - \lambda_1 t_2), & if\ 2\lambda \neq \lambda_1 \quad (2a) \\ (2\lambda_c)^{2n} \exp(-2\lambda_c (t_1 + t_2)), & if\ 2\lambda = \lambda_1 \quad (2b) \end{cases}$$

where $t_i = \sum_{j=1}^{n} y_{ij}$, $i = 1, 2$.

The MLE's of $\lambda_i$ and $\lambda_c$ are given, respectively, by

$$\hat{\lambda}_i = \frac{n}{(2-i)T_{i+1}}, \qquad i = 0,1$$

$$\hat{\lambda}_c = \frac{n}{T_1 + T_2} \tag{3}$$

and the MLE of $R(t)$ can be easily obtained, where $\lambda_0 = \lambda$.

Let $G(n,b)$ denote the gamma probability density function(pdf) of the form

$$f_{n,b}(x) = \frac{b^n}{\Gamma(n)} x^{n-1} e^{-bx}, \qquad x \geq 0.$$

We note that $T_i \sim G(n,(3-i)\lambda_{i-1})$ when $2\lambda \neq \lambda_1$, and $T_1 + T_2 \sim G(2n, 2\lambda_c)$ when $2\lambda = \lambda_1$. It can be shown that

$$\begin{aligned} E(\hat{\lambda}_i) &= \frac{n}{n-1}\lambda_i, \quad n>1, & Var(\hat{\lambda}_i) &= \frac{n^2}{(n-2)(n-1)^2}\lambda_i^2, \quad n>2 \\ E(\hat{\lambda}_c) &= \frac{2n}{2n-1}\lambda_c, & Var(\hat{\lambda}_c) &= \frac{2n^2}{(n-1)(2n-1)^2}\lambda_c^2, \quad n>1 \end{aligned} \tag{4}$$

and higher moments can also be readily obtained. From (4), we know that the MLE's of $\lambda_i'$s and $\lambda_c$ are positively biased. Since $(T_1, T_2)$ and $T + T_2$ are complete sufficient statistics for the family of distributions (2a) and (2b), respectively. The UMVUE of $\lambda_i'$s and $\lambda_c$ are obtained as

$$\begin{aligned} \tilde{\lambda}_i &= \frac{n-1}{(2-i)T_{i+1}}, \quad n>1, \quad i=0,1, \\ \tilde{\lambda}_c &= \frac{2n-1}{2(T_1+T_2)}, \quad n>1. \end{aligned} \tag{5}$$

We now find the OMLE of the reliability function when $2\lambda \neq \lambda_1$. Naturally, failure of one component reduces the additional mean life of the remaining component by increasing $\lambda$ to $\lambda_1$. Our problem then becomes one of isotonic estimations, that is, the estimation of $R(t)$ subject to the order restriction $\lambda \leq \lambda_1$. When certain ordering is known a priori about the parameters to be estimated, estimation under order restriction is required to reflect this knowledge.

Order restricted inference has been worked by Marshall & Proschan(1965), Barlow et al.(1972) and Kaur & Singh(1991). Kaur & Singh(1991) considered maximum likelihood estimation of two ordered exponential means and established that MLE's obtained under the order restriction have pointwise smaller mean squared error than the usual estimators, i.e., the sample means. By using the max-min formula of isotonic regression(see Barlow et al.(1972)), we have the OMLE's of $\lambda$ and $\lambda_1$ as

$$\hat{\hat{\lambda}} = \min\left\{\frac{n}{2T_1}, \frac{2n}{2T_1 + T_2}\right\},$$

and

$$\hat{\hat{\lambda}}_1 = \max\left\{\frac{n}{T_2}, \frac{2n}{2T_1 + T_2}\right\} \tag{6}$$

Substituting $(\hat{\hat{\lambda}}, \hat{\hat{\lambda}}_1)$ for $(\lambda, \lambda_1)$ in (1a), we can obtain the OMLE of $R(t)$.

The UMVUE of $R(t)$ is given in the following theorem.

**Theorem 1.** Let $U = \min(T_1, T_2)$, $V = \max(T_1, T_2)$, and $W = T_1 + T_2$. Then, for $n > 1$, the UMVUE of $R(t)$ is given by

(i) If $\lambda \neq 2\lambda_1$;

$$\tilde{R}(t) = \begin{cases} 1, & \text{if } t = 0, \\ \left(1 - \frac{t}{U}\right)^{n-1} + s(t), & \text{if } 0 < t \leq U, \\ s(U), & \text{if } U < t \leq V, \\ s(U) - s(t - V), & \text{if } V < t \leq U + V, \\ 0, & \text{if } U + V < t, \end{cases} \tag{7}$$

where $s(x) = (n - 1/U)\int_0^x (1 - t - w/V)^{n-1}(1 - w/U)^{n-2}\,dw.$

(ii) If $\lambda = 2\lambda_1$;

$$\tilde{R}(t) = \begin{cases} 1, & \text{if } t = 0, \\ \left(1 - \frac{t}{W}\right)^{2n-1} + \frac{(2n-1)t}{W} \cdot \left(1 - \frac{t}{W}\right)^{2n-2}, & \text{if } t < W, \\ 0, & \text{if } W \leq t. \end{cases} \tag{8}$$

Proof of part (i)

Define

$$\phi_t(Y_{11}, Y_{21}) = \begin{cases} 1, & \text{if } Y_{11} + Y_{21} > t, \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

Then $\phi_t(Y_{11}, Y_{21})$ is an unbiased estimator of $R(t)$. Therefore, by the Rao-Blackwell and Lehmann-Scheffe theorems, the UMVUE of $R(t)$ is given by

$$\tilde{R}(t) = E[\phi_t(Y_{11}, Y_{21}) | T_1 = t_1, T_2 = t_2] \\ = \iint_R f_1(y_{11}|t_1) \cdot f_2(y_{21}|t_2) dy_{11} dy_{21} \qquad (10)$$

where $R = \{(y_{11}, y_{21}); y_{11} + y_{21} > t, 0 < y_{i1} < t_i, i = 1,2\}$ and $f_i(y_{i1}|t_i)$ is the conditional pdf of $y_{i1}$ given $t_i$. It can be seen that

$$f_i(y_{i1}|t_i) = \frac{n-1}{t_i}\left(1 - \frac{y_{i1}}{t_i}\right)^{n-2}, \qquad 0 < y_{i1} < t_i \qquad (11)$$

Substituting (11) into (10), we obtain the stated result. The proof of part (ii) can be easily shown by using the fact that $W \sim G(2n, 2\lambda_c)$ . □

**Table 1.** Estimated Bias and MSE for $\lambda = 1$

| | | | Bias | | | MSE | | |
|---|---|---|---|---|---|---|---|---|
| $\lambda_1$ | $R(t)$ | n | MLE | OMLE | UMVUE | MLE | OMLE | UMVUE |
| 1.0 | .99094 | 5 | -.00399 | -.00343 | .00020 | .00010 | .00008 | .00004 |
| | | 10 | -.00177 | -.00156 | .00008 | .00003 | .00003 | .00002 |
| | | 15 | -.00131 | -.00120 | -.00011 | .00002 | .00002 | .00001 |
| | | 20 | -.00096 | -.00087 | -.00008 | .00000 | .00000 | .00000 |
| | | 30 | -.00064 | .00059 | -.00007 | .00000 | .00000 | .00000 |
| 1.3 | .98834 | 5 | -.00504 | -.00466 | .00025 | .00016 | .00014 | .00007 |
| | | 10 | -.00225 | -.00216 | .00010 | .00004 | .00004 | .00003 |
| | | 15 | -.00167 | -.00163 | -.00014 | .00003 | .00003 | .00002 |
| | | 20 | -.00123 | -.00120 | -.00011 | .00002 | .00002 | .00001 |
| | | 30 | -.00082 | -.00081 | -.00009 | .00001 | .00001 | .00000 |
| 1.5 | .98664 | 5 | -.00571 | -.00541 | .00028 | .00020 | .00019 | .00009 |
| | | 10 | -.00256 | -.00250 | .00011 | .00006 | .00006 | .00004 |
| | | 15 | -.00190 | -.00188 | -.00016 | .00004 | .00004 | .00003 |
| | | 20 | -.00140 | -.00138 | -.00012 | .00002 | .00002 | .00002 |
| | | 30 | -.00093 | -.00093 | -.00010 | .00001 | .00001 | .00001 |
| 3.0 | .97456 | 5 | -.01003 | -.00999 | .00049 | .00063 | .00062 | .00030 |
| | | 10 | -.00457 | -.00456 | .00020 | .00018 | .00018 | .00012 |
| | | 15 | -.00341 | -.00341 | -.00030 | .00012 | .00012 | .00009 |
| | | 20 | -.00252 | -.00252 | -.00024 | .00007 | .00007 | .00005 |
| | | 30 | -.00168 | -.00168 | -.00018 | .00005 | .00005 | .00004 |
| 5.0 | .96020 | 5 | -.01427 | -.01427 | .00069 | .00131 | .00131 | .00069 |
| | | 10 | -.00660 | -.00660 | .00028 | .00040 | .00040 | .00027 |
| | | 15 | -.00495 | -.00495 | -.00045 | .00027 | .00027 | .00020 |
| | | 20 | -.00370 | -.00370 | -.00037 | .00016 | .00016 | .00013 |
| | | 30 | -.00245 | -.00245 | -.00027 | .00010 | .00010 | .00009 |

## 3. Numerical Comparisons

In this section, we present some numerical results and compare the bias and the mean square error(MSE) of the UMVUE, OMLE and MLE of $R(t)$ for moderate sample size. Samples of size $n$ were generated. The experiment was repeated 1000 times with $\lambda_1/\lambda = 1.0, 1.3, 1.5, 3.0, 5.0$ for $\lambda = 1,2$ and $t = 0.1$. For each repetition the estimated bias and MSE were calculated for each estimator. The results are given in Tables 1 and 2.

In general, as expected, the estimated bias of the UMVUE is found to be considerably smaller than the estimated biases of the MLE and OMLE. The estimated MSE's of the three estimators of $R(t)$ tend to increase as the ratio of $\lambda_1$ to $\lambda$ increases, and to decrease when $n$ increases. Also, the estimated biases and MSE's of the MLE and OMLE are comparable and the differences in bias and MSE are found to be quite small and tend to zero as the ratio of $\lambda_1$ to $\lambda$ increases.

**Table 2.** Estimated Bias and MSE for $\lambda = 2$

| | | | Bias | | | MSE | | |
|---|---|---|---|---|---|---|---|---|
| $\lambda_1$ | $R(t)$ | n | MLE | OMLE | UMVUE | MLE | OMLE | UMVUE |
| 2.0 | .96714 | 5 | -.01252 | -.01067 | .00052 | .00072 | .00066 | .00031 |
| | | 10 | -.00547 | -.00506 | .00039 | .00027 | .00025 | .00018 |
| | | 15 | -.00382 | -.00346 | -.00000 | .00017 | .00017 | .00013 |
| | | 20 | -.00305 | -.00289 | -.00022 | .00011 | .00010 | .00008 |
| | | 30 | -.00191 | -.00180 | -.00006 | .00006 | .00006 | .00005 |
| 2.6 | .95812 | 5 | -.01553 | -.01430 | .00055 | .00111 | .00106 | .00051 |
| | | 10 | -.00685 | -.00673 | .00043 | .00043 | .00042 | .00029 |
| | | 15 | -.00476 | -.00468 | -.00002 | .00028 | .00028 | .00021 |
| | | 20 | -.00380 | -.00378 | -.00028 | .00017 | .00017 | .00014 |
| | | 30 | -.00238 | -.00237 | -.00007 | .00010 | .00010 | .00008 |
| 3.0 | .95231 | 5 | -.01737 | -.01646 | .00055 | .00140 | .00136 | .00065 |
| | | 10 | -.00771 | -.00767 | .00043 | .00055 | .00055 | .00038 |
| | | 15 | -.00534 | -.00533 | -.00004 | .00035 | .00036 | .00027 |
| | | 20 | -.00426 | -.00425 | -.00032 | .00022 | .00022 | .00017 |
| | | 30 | -.00266 | -.00266 | -.00008 | .00012 | .00012 | .00011 |
| 6.0 | .91334 | 5 | -.02801 | -.02791 | .00007 | .00380 | .00379 | .00202 |
| | | 10 | -.01292 | -.01292 | .00018 | .00163 | .00163 | .00119 |
| | | 15 | -.00878 | -.00878 | -.00026 | .00104 | .00104 | .00084 |
| | | 20 | -.00691 | -.00691 | -.00057 | .00064 | .00064 | .00053 |
| | | 30 | -.00431 | -.00431 | -.00015 | .00036 | .00036 | .00032 |
| 10.0 | .87195 | 5 | -.03637 | -.03642 | -.00129 | .00680 | .00681 | .00410 |
| | | 10 | -.01736 | -.01736 | -.00067 | .00314 | .00314 | .00243 |
| | | 15 | -.01153 | -.01153 | -.00065 | .00198 | .00198 | .00167 |
| | | 20 | -.00889 | -.00889 | -.00080 | .00122 | .00122 | .00105 |
| | | 30 | -.00554 | -.00554 | -.00019 | .00067 | .00067 | .00061 |

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