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Interval Estimations for Reliablility in Stress-Strength Model by Bootstrap Method

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Abstract We construct the approximate bootstrap confidence intervals for reliability (R) when the distributions of strength and stress are both normal. Also we propose percentile, bias correct (BC), bias correct acceleration (BCa), and percentile-t intervals for R. We compare with the accuracy of the proposed bootstrap confidence intervals and classical confidence interval based on asymptotic normal distribution through Monte Carlo simulation. Results indicate that the confidence intervals by bootstrap method work better than classical confidence interval. In particular, confidence intervals by BC and BCa method work well for small sample and/or large value of true reliability.

1. Introduction

The stress-strength model has been widely used in a variety of areas including estimating for the reliability of a design procedures. Let X be the strength of the component and Y be the stress placed on the component by the operating environment in the stress-strength model. In many applications, the stress distribution may be known to the investigator. Church and Hariss(1970) obtained an estimator and an approximate confidence interval for estimator estimator $R = P(Y < X)$ based on asymptotic normal distribution when the distribution of stress is known. Since the true distribution of the estimator of R is often skewed and biased for small sample size, the interval based on the asymptotic normal distribution may deteriorate the accuracy. We will use the bootstrap method in order to avoid these problems. Efron(1979) initially introduced the bootstrap method to assign the accuracy for an estimator. In particular, Efron (1981,1982, 1987) and Hall(1988) proposed the percentile method, the bias correct method

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(BC method), the bias correct acceleration method(BCa method), and the percentile-t method to construct approximate confidence interval for an estimator.

In this paper, we derive the large sample property for the bootstrap estimator of R. Also we propose approximate confidende intervals for R based on percentile, BC, BCa, and percentile-t methods. We investigate the accuracy of the proposed bootstrap confidence intervals and confidence interval based on asymptotic normal distribution through Monte Carlo simulation. In particular, we observe the accuracy of these intervals for small sample size and/or large value of R.

2. Consistency for Bootstrap Estimator

The bootstrap method is a resampling scheme which one attempts to learn the sampling properties of a statistic by recomputing its value on the basis of a new sample realized from the original one. The bootstrap method provides confidence interval estimates by using the plug-in principle for R.

Without loss of generality, we assume that known stress distribution function $G(y)$ is the standard normal distribution function $\Phi(y)$ and that the strength distribution $F(x)$ is the normal distribution function with mean μ and variance σ^2. Then the reliability is computed as $R = \Phi(\theta)$, where $\theta = \frac{\mu}{\sqrt{1+\sigma^2}}$. Let $\underline{X}_n$ = $(X_1, X_2, \cdots, x_n)$ be the random sample of size n from the strength distribution function $F(x)$. Then Church and Harris (1970) obtained the following estimator $\hat{R}$ of R given by

$$\hat{R} = \Phi\left(\frac{\overline{X}}{\sqrt{1+\hat{\sigma}^2}}\right) \tag{1}$$

where $\overline{X} = (1/n)\sum_{i=1}^{n} X_i$, $\hat{\sigma}^2 = \frac{1}{(n-1)}\sum_{i=1}^{n}(X_i - \overline{X})^2$, and $\hat{\theta} = \frac{\overline{X}}{\sqrt{1+\hat{\sigma}^2}}$

The bootstrap procedure for construction of confidence interval for R may be described as follows:

(1) Compute the plug-in estimates of μ and σ^2 given by $\overline{X}$ and $S^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2$ from $\underline{X}_n$. And construct the sampling distribution F_n (from the $\underline{X}_n$) based on $\overline{X}$ and S^2. That is, $F_n \sim N(\overline{X}, S^2)$.

(2) Generate B random samples of size n from fixed F_n. The corresponding sample called the *bootstrap sample* is denoted by

$\underline{X}_n^{*b} = (X_1^{*b}, X_2^{*b}, \cdots, X_n^{*b}),\quad b = 1, 2, \cdots, B$.

(3) Construct the bootstrap estimators of R, $\hat{R}^{*b} = \Phi(\hat{\theta}^{*b})$, where $\hat{\theta}^{*b} = \dfrac{\overline{X}^{*b}}{\sqrt{1+S^{2*b}}}$, $\overline{X}^{*b} = \dfrac{1}{n}\sum_{i=1}^{n} X_i^{*b}$, and $S^{2*b} = \dfrac{1}{n}\sum_{i=1}^{n}(X_i^{*b} - \overline{X}^{*b})^2$. We call $\overline{X}^{*b}$ and S^{2*b} by *bootstrap estimators* for μ and σ^2, respectively.

The following theorem is related to the weak law of large numbers and the convergence in distribution of bootstrap estimator.

Theorem Suppose that $\underline{X}_n$ is the random sample of size n from the strength distribution function $F(x)$. And suppose that $\underline{X}_n^*$ is the bootstrap sample of size n from the sample distribution function F_n. Then the bootstrap estimator $\hat{R}^*$ of R is a consistent estimator of R.

Proof. For arbitrary positive ε,

$$\begin{aligned}
P(|\overline{X}^* - \overline{X}| \geq \varepsilon) &\leq \frac{E(\overline{X}^* - \overline{X})^2}{\varepsilon^2} \\
&= \frac{E\left[E[(\overline{X}^* - \overline{X})^2 | \underline{X}_n]\right]}{\varepsilon^2} \\
&= \frac{E(S^2/n)}{\varepsilon^2} \\
&= \frac{(n-1)\sigma^2}{\varepsilon^2 n^2} \to 0,\ \text{as } n \to \infty.
\end{aligned}$$

And

$$\begin{aligned}
P(|S^{2*} - S^2| \geq \varepsilon) &\leq \frac{E(S^{2*} - S^2)^2}{\varepsilon^2} \\
&= \frac{E\left[E\left[(S^{2*} - S^2)^2 | \underline{X}_n\right]\right]}{\varepsilon^2} \\
&= \frac{(2n-1)}{\varepsilon^2 n^2} E(S^4) \\
&= \frac{(2n-1)(n^2-1)\sigma^4}{\varepsilon^2 n^4} \to 0,\ \text{as } n \to \infty.
\end{aligned}$$

Therefore, $\overline{X}^*$ and S^{2*} converge in probability to μ and σ^2, respectively. Hence,

$\hat{\theta}^* = \dfrac{\overline{X}^*}{\sqrt{1+S^{2*}}}$ is also a consistent estimator of θ. Since Φ is continuous function, $\hat{R}^* = \Phi(\dfrac{\overline{X}^*}{\sqrt{1+S^{2*}}})$ is a consistent estimator of R.

Note that, under the assumptions of theorem, the asymptotic distribution of $\hat{R}^*$ and R are same.

3. Bootstrap Confidence Intervals for Reliability

In this Section we compare approximate confidence interval based on asymptotic normal distribution and approximate bootstrap confidence intervals for R. All confidence intervals are two-sided and equal-tailed with confidence level $100(1-2\alpha)\%$.

Church and Harris's method

Church and Harris(1970) proved that $\hat{\theta}$ has asymptotic normal distribution with mean θ and variance

$$\sigma_{\hat{\theta}}^2 = \frac{\sigma^2}{1+\sigma^2}\left(\frac{1}{n}+\frac{\theta^2\sigma^2}{2(n-1)(1+\sigma^2)}\right).$$

The asymptotic variance of $\hat{\theta}$ is estimated by

$$\hat{\sigma}_{\hat{\theta}}^2 = \frac{\hat{\sigma}^2}{1+\hat{\sigma}^2}\left(\frac{1}{n}+\frac{\hat{\theta}^2\hat{\sigma}^2}{2(n-1)(1+\hat{\sigma}^2)}\right).$$

Hence, $100(1-2\alpha)\%$ confidence interval for R based on the asymptotic normal distribution is given by

$$\left(\hat{R}+z^{(\alpha)}\hat{\sigma}_{\hat{\theta}}\phi(\hat{\theta}),\qquad \hat{R}+z^{(1-\alpha)}\hat{\sigma}_{\hat{\theta}}\phi(\hat{\theta})\right) \tag{2}$$

where $z^{(1-\alpha)}$ is the $100(1-\alpha)$ percentile of standard normal distribution and $\phi(\cdot)$ is the standard normal probability density function.

Percentile method

The confidence interval by the bootstrap percentile method (percentile interval) is obtained by percentiles of the empirical bootstrap distribution of bootstrap estimator $\hat{R}^*$. Let $\hat{H}$ be the empirical cumulative distribution function of $\hat{R}^*$.

Then it is constructed by $\hat{H}(s)=\frac{1}{B}\sum_{b=1}^{B} I(\hat{R}^{*b}\le s)$, where s is arbitrary real value and $I(\cdot)$ is an indicator function. And let $\hat{H}^{-1}(\alpha)$ be the 100α empirical percentile of $\hat{R}^{*}$ given by

$$\hat{H}^{-1}(\alpha)=\inf\{s:\hat{H}(s)\ge\alpha\} \tag{3}$$

That is, $\hat{H}^{-1}(\alpha)$ is the $B\alpha$th value in the odered list of the B replications of $\hat{R}^{*b}$. If $B\alpha$ is not an integer, we can take the largest integer that less than or equal to $(B+1)\alpha$. Then $100(1-2\alpha)\%$ percentile interval for R is approximated by

$$\left(\hat{H}^{-1}(\alpha),\qquad \hat{H}^{-1}(1-\alpha)\right) \tag{4}$$

Bias correct method

The BC method adjusts a possible bias in estimating R. The bias correction is given by

$$\hat{z}_0=\Phi^{-1}\left(\hat{H}(\hat{R})\right)=\Phi^{-1}\left[\frac{1}{B}\sum_{b=1}^{B} I(\hat{R}^{*b}\le\hat{R})\right],$$

where $\Phi^{-1}(\cdot)$ indicates the inverse function of the standard normal cumulative distribution function. $\hat{z}_0$ is the discrepancy between the medians of $\hat{R}^{*}$ and $\hat{R}$ in normal unit. Therefore, we have $100(1-2\alpha)\%$ approximate BC interval for R given by

$$\left(\hat{H}^{-1}(\alpha_1),\qquad \hat{H}^{-1}(\alpha_2)\right) \tag{5}$$

where $\alpha_1=\Phi(2\hat{z}_0+z^{(\alpha)})$ and $\alpha_2=\Phi(2\hat{z}_0+z^{(1-\alpha)})$.

Bias correct acceleration method

The BCa method corrects both the bias and standard error for $\hat{R}$. The confidence interval by BCa method(BCa interval) requires to calculate the bias-correction constant $\hat{z}_0$ and the acceleration constant $\hat{a}$. In fact, the bias-correction constant $\hat{z}_0$ is the same as that of BC method. And $\hat{a}$, measured on a normalized scale, refers to the rate of change of the standard error of $\hat{R}$ with respect to the true reliability R.

For the parametric bootstrap method, all calculations relate only to the sufficient statistic $E_1=\frac{1}{n}\sum_{i=1}^{n}X_i$ and $E_2=\frac{1}{n}\sum_{i=1}^{n}(X_i-\overline{X})^2$ for μ and σ^2. Of course, E_1 is distributed $N(\mu,\sigma^2/n)$ and E_2 is distributed $(\sigma^2/n)\cdot\chi^2(n-1)$.

Also, E_1 and E_2 are independent. Let $\underline{E}' = (E_1, E_2)$ and $\underline{\eta}' = (\mu, \sigma^2)$. Then the joint probability density function of $\underline{E}$ can be written as

$$f_{\underline{\eta}'}(\underline{E}') = f_0(\underline{E}) \exp\left[g_0(\underline{\eta}'\underline{E}) - \Psi_0(\underline{\eta}') \right],$$

where $f_0(\underline{E}') = \dfrac{n^{n/2}}{\sqrt{2\pi}\ \Gamma((n-1)/2) 2^{(n-1)/2}}$,

$g_0(\underline{\eta}'\underline{E}) = -\dfrac{nE_1^2 - 2nE_1\mu + nE_1}{2\sigma^2}$,

and $\Psi_0(\underline{\eta}') = \dfrac{n\mu^2}{2\sigma^2} + \dfrac{n}{2}\log(\sigma^2)$.

For multiparameter family case, we will find $\hat{a}$ following Stein's construction(1956). That is, We replace the multiparameter family $\Im = \{f_{\underline{\eta}'}(X)\}$ by the least favorable one parameter family $\hat{\Im} = \{\hat{f}_{\underline{\eta}'}(X) \equiv f_{\underline{\hat{\eta}}' + \lambda \underline{\hat{\omega}}}(X)\}$. Then we first obtain $\underline{\hat{\omega}}$ such that the least favorable direction at $\underline{\eta} = \underline{\hat{\eta}}$ is defined to be $\underline{\hat{\omega}} = (\ddot{l}_{\underline{\hat{\eta}}})^{-1} \hat{\nabla}_{\underline{\hat{\eta}}'}$ where $\underline{\hat{\eta}} = (E_1, E_2)'$, $\ddot{l}_{\underline{\hat{\eta}}}$ is Fisher information matrix, and $\hat{\nabla}_{\underline{\hat{\eta}}}$ is the gradient of R given by $\hat{\nabla}_{\underline{\hat{\eta}}} = \dfrac{\partial R}{\partial \underline{\eta}}\Big|_{\underline{\eta} = \underline{\hat{\eta}}}$. By some algebraic calculation, we have

$$\ddot{l}_{\underline{\hat{\eta}}} = \begin{bmatrix} n/E_0 & 0 \\ 0 & n/E_0^2 \end{bmatrix}$$

$$\hat{\nabla}_{\underline{\hat{\eta}}} = \begin{bmatrix} \dfrac{1}{\sqrt{1+E_2}} \phi\left(\dfrac{E_1}{\sqrt{1+E_2}}\right) \\ -\dfrac{E_1}{2(1+E_2)^{3/2}} \phi\left(\dfrac{E_1}{\sqrt{1+E_2}}\right) \end{bmatrix}$$

Hence, we have

$$\underline{\hat{\omega}} = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix},$$

where $W_1 = \dfrac{E_2}{n\sqrt{1+E_2}} \phi\left(\dfrac{E_1}{\sqrt{1+E_2}}\right)$

and $W_2 = -\dfrac{nE_2^2}{n(1+E_2)^{3/2}} \phi\left(\dfrac{E_1}{\sqrt{1+E_2}}\right)$.

By the method of Efron(1987), $\hat{a}$ can be obtained as the following

$$\hat{a} = \frac{1}{6\sqrt{n}} \frac{\hat{\Psi}^{(3)}(0)}{\left(\hat{\Psi}^{(2)}(0)\right)^{(3/2)}} \tag{6}$$

where $\hat{\Psi}^{(j)}(0) = \frac{\partial^j \Psi_0(\hat{\underline{\eta}} + \lambda \hat{\underline{\omega}})}{\partial \lambda^j} |_{\lambda=0}$.

Calculating $\hat{\Psi}^{(j)}(.)$ and $\hat{\underline{\omega}}$, we can obtain $\hat{a}$ by

$$\hat{a} = \frac{\sqrt{2}}{6n} \frac{E_2^{1/2}(6W_1^2E_2^2W_2 - 12E_1W_1E_2W_2^2 + 6E_1^2W_2^3 - 2E_2W_2^3)}{(W_2^2E_2 - 2W_1^2E_2^2 + 4E_1W_1E_2W_2 - 2E_1^2W_2^2)^{3/2}} \tag{7}$$

Therefore, we have $100(1-2\alpha)\%$ approximate BCa interval for R by

$$\left(\hat{H}^{-1}(\alpha_3), \qquad \hat{H}^{-1}(\alpha_4)\right) \tag{8}$$

where $\alpha_3 = \Phi\left[\hat{z}_0 + \frac{\hat{z}_0 + z^{(\alpha)}}{1 - \hat{a}(\hat{z}_0 + z^{(\alpha)})}\right]$, $\alpha_4 = \Phi\left[\hat{z}_0 + \frac{\hat{z}_0 + z^{(1-\alpha)}}{1 - \hat{a}(\hat{z}_0 + z^{(1-\alpha)})}\right]$,

and $\hat{z}_0$ is the same as that of BC method.

Percentile-*t* method

The confidence interval by the bootstrap percentile-*t* method(percentile-*t* interval) is constructed by using the bootstrap distribution of an approximately pivotal quantity for $\hat{R}$ instead of the bootstrap distribution of $\hat{R}$. We define an approximate bootstrap pivotal quantity for R by

$$\hat{R}^*{}_{STUD} = \frac{\hat{R}^* - \hat{R}}{\phi(\hat{\theta}^*)\hat{\sigma}^*{}_{\hat{\theta}}} \tag{9}$$

where $\hat{\sigma}^*{}_{\hat{\theta}}$ is the bootstrap estimator of $\hat{\sigma}_{\hat{\theta}}$, that is,

$$\hat{R}^*{}_{STUD} = \frac{\hat{R}^* - \hat{R}}{\phi(\hat{\theta}^*)\hat{\sigma}^*{}_{\hat{\theta}}} \tag{10}$$

We compute the empirical distribution function $\hat{H}_{STUD}$ of $\hat{R}^*_{STUD}$. by

$$\hat{H}_{STUD}(s) = \frac{1}{B}\sum_{b=1}^{B} I(\hat{R}^{*b}_{STUD} \le s) \tag{11}$$

for all s.

Let $\hat{H}^{-1}_{STUD}(\alpha)$ denote 100α empirical percentile of $\hat{R}^*_{STUD}$. And we compute $\hat{H}^{-1}_{STUD}(\alpha)$ by

$$\hat{H}^{-1}_{STUD}(\alpha) = \inf\{s: \hat{H}_{STUD}(s) \geq \alpha\} \tag{12}$$

That is, $\hat{H}^{-1}_{STUD}(\alpha)$ is the $B\alpha$th value in the ordered list of the B replications of $\hat{R}^*_{STUD}$. Then we have $100(1-2\alpha)\%$ approximate percentile-*t* interval for *R* by

$$\left(\hat{R} + \hat{\sigma}_{\hat{\theta}}\phi(\hat{\theta})\hat{H}^{-1}_{STUD}(\alpha), \qquad \hat{R} + \hat{\sigma}_{\hat{\theta}}\phi(\hat{\theta})\hat{H}^{-1}_{STUD}(1-\alpha)\right) \tag{13}$$

4. Comparisons and Conclusions

To compare the approximate confidence interval estimates, we first will compute the results obtained in section 3. The methods are compared based mainly on coverage probability (CP_1), length (L_1), and interval shape (SP_1). Let $\hat{R}_{lo}$ and $\hat{R}_{up}$ be the lower and upper limit of confidence interval. Then SP_1 is defined by

$$SP_1 = \frac{\hat{R}_{up} - \hat{R}}{\hat{R} - \hat{R}_{lo}} \tag{14}$$

If $SP_1 > 1.00$, then it indicates greater distance from $\hat{R}_{up}$ to $\hat{R}$ than from $\hat{R}$ to $\hat{R}_{lo}$. That is, empirical distribution of $\hat{R}$ has a right skewness. If $SP_1 < 1.00$, then it indicates a left skewness. If $SP_1 = 1.00$, then the interval is symmetric about $\hat{R}$. The normal random numbers were generated by IMSL subroutine RNNOF. We use that the true reliabilities of R are 0.3,0.5,0.7, and 0.9 and that sample sizes n are 5,10,20, and 50. The used confidence level $(1-2\alpha)$ is 0.90. For given independent random samples the approximate bootstrap confidence intervals were constructed by each method with bootstrap replications $B = 1000$ times. And the Monte Carlo samplings were repeated 500 times.

We can note the following properties through Figures 1-6.

(1) In the cases of small sample, the proposed approximate bootstrap confidence intervals have better accuracy than the interval based on asymptotic normal distribution except for the percentile-*t* interval. In particular, BC and BCa intervals work well for all R.

(2) For most cases of R, the percentile, BC, and BCa intervals also work better

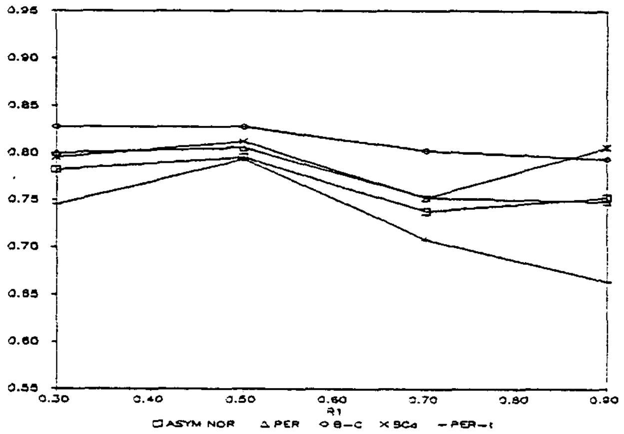

Figure 1. Plot of CP_1 against R when n=5

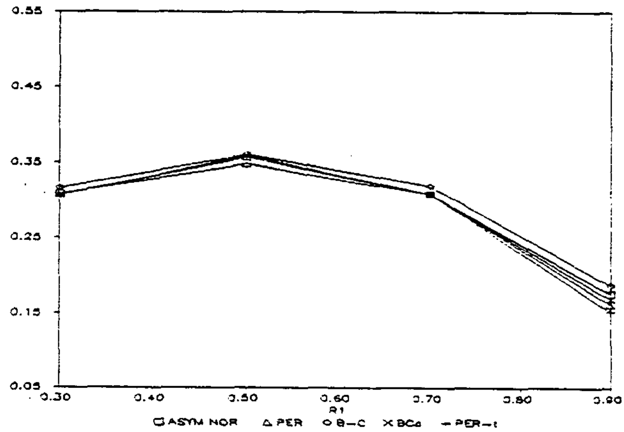

Figure 2. Plot of L_1 against R when n=5

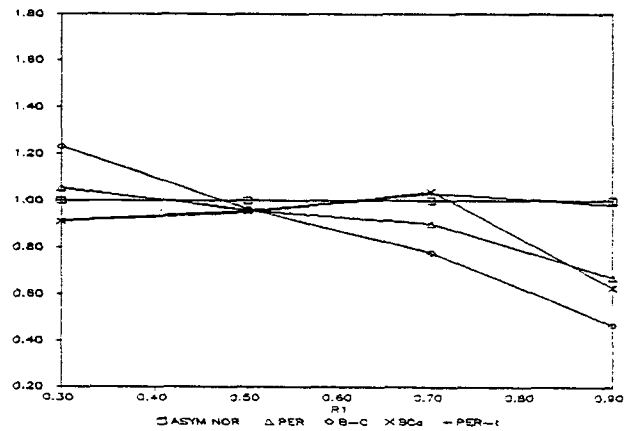

Figure 3. Plot of SP_1 against R when n=5

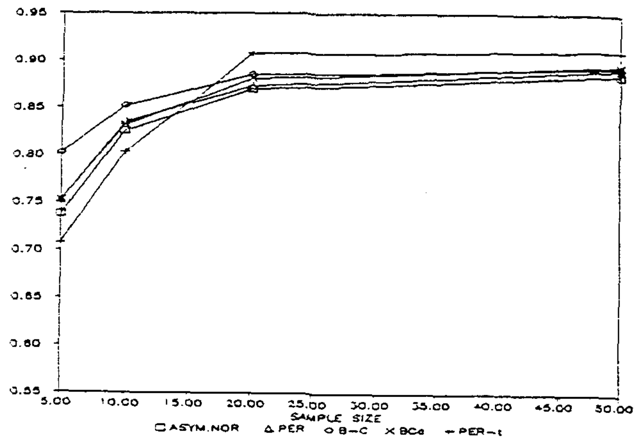

Figure 4. Plot of CP_1 against n when R=0.7

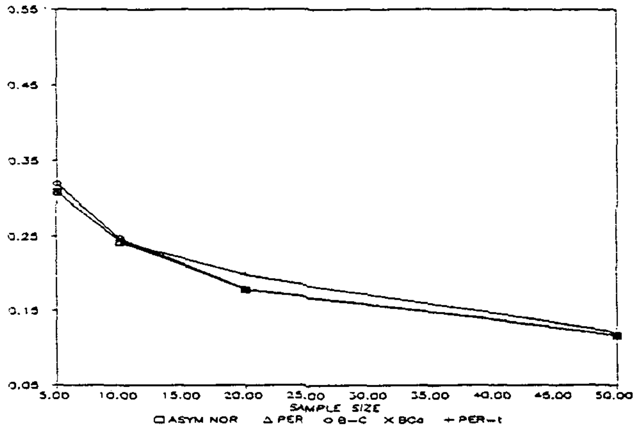

Figure 5. Plot of L_1 against n when R=0.7

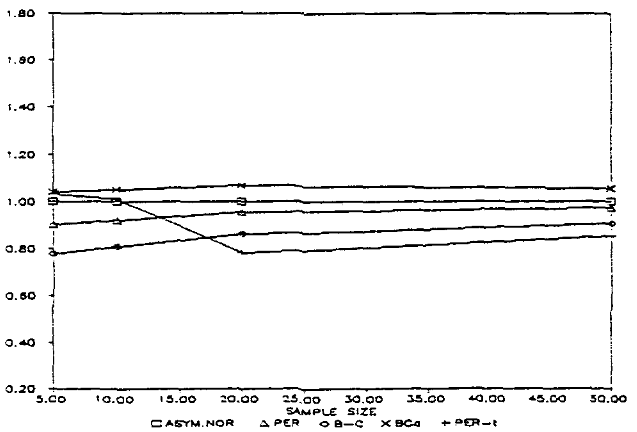

Figure 6. Plot of SP_1 against n when R=0.7

than the interval based on the asymptotic normal distribution.

(3) The L_1's of all approximate confidence intervals tend to decrease as R deviates from 0.5. As a whole, the L_1's of the intervals by all methods are nearly same.

(4) The SP_1's of the all approximate intervals by the bootstrap methods tend to decrease as R increase. That is, the proposed intervals tend to be left-skewness as R increase. The SP_1 of BC interval tends to be quick left skewed for R.

(5) The CP_1's of all approximate intervals converge to true coverage level. The CP_1's of the percentile, BC, and BCa intervals work better than that of the interval by asymptotic normality for all sample sizes.

(6) The L_1's of the approximate intervals by all methods converge to true interval length.

(7) All approximate intervals by the bootstrap methods are left-skewed. But the intervals by all methods tend slowly to be symmetric for R.

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