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Reconstruction of the Wave Speed and Density from Reflection Coefficients by Downward Continuation Algorithm

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ABSTRACT: The purpose of this paper is recovery of the profiles of the wave speed and density from the reflection coefficients of the continuously layered acoustic medium with depth dependent density and wave speed at various angles of incidence. A downward continuation or layer stripping algorithm, which recursively reconstructs the medium in increasing depth and then strips away the effects of the reconstructed portion of the medium, is the method with fewer computations than integral equation procedures. This paper implements an improved downward continuation algorithm that uses reflection data at several angles of incidence and performs a least-squares fitting at each depth. The result is a considerable improvement in performance over the usual downward continuation algorithm.

INTRODUCTION

The seismic reflection method of exploration is used to map the configuration and nature of remote and inaccessible rock layers beneath the subsurface of the earth (Silvia and Robinson, 1979). In this technique, a seismic disturbance is generated at or near the surface of the earth, and the response of the earth to this disturbance due to the changes in the acoustic impedance of the subsurface layering is recorded on the surface at some distance away (Telford *et al.*, 1990; Shon and Yamamoto, 1992).

The goal of this paper is to reconstruct the wave speed $c(z)$ and density $\rho(z)$ profiles from the measured reflection responses. In this paper, the acoustic medium is supposed to be laterally homogeneous, and it is characterized by its density $\rho(z)$ and wave speed $c(z)$ profiles as a function of depth z. This problem has been formulated as an inverse scattering problem by Ware and Aki (1969) and Coen (1981). This problem was transformed into a normal-incidence inverse problem, which requires the solution of two Marchenko integral equations for two acoustic impedance profiles as functions of two different travel times. These two profiles are then analytically inverted to yield the $\rho(z)$ and $c(z)$ profiles. However, this approach requires lots of computation involved in the solution of two integral equations and the travel-time inversions. In addition, the profile inversions is extremely unstable in the presence of noise.

Downward continuation or layer stripping algorithm is an alternative to the integral equation approaches for solving inverse scattering problems (Claerbout *et al.*, 1984; Yagle *et al.*, 1984; Bruckstein *et al.*, 1985; Carrion *et al.*, 1985; Santosa *et al.*, 1985). A downward continuation algorithm works by recursively reconstructing the subsurface physical parameters and stripping away the effects of the medium at each depth. The waves are decomposed into upgoing and downgoing waves, and at each depth portions of each wave are scattered into the other wave. The amount of scattering at a given depth is characterized by a reflectivity function $r(z)$. The reflectivity function includes primaries and all multiples. By causality, the first reflection of the downgoing wave into the upgoing wave at each depth is due solely to the reflectivity function at that depth. Therefore, the reflectivity function at that depth can be determined, and the waves propagates deeper into the medium. The advantage of using a downward continuation algorithm is that it requires fewer computations than solving the integral equations. Discretizing the medium into N layers and solving the resulting discretized integral equation by Gaussian elimination requires $O(N^3)$ computations, while the downward continuation algorithm for solving the same problem requires only $O(N^2)$ computations.

To reconstruct the density and wave speed profiles this paper implements a improved downward continuation algorithm that uses reflection coefficients at different angles of incidences and performs a least-squares fitting at each depth. This results in considerable improvement in performance.

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NUMERICAL FORMULATION

An impulsive plane wave is incident at an angle θ from the vertical to the surface, on a continuously layered 2D medium. The reflection coefficient $R(\omega, \theta)$ at the interface between the medium and an overlying homogeneous infinite half-space (air or ocean water) is measured. The medium has unknown depth-dependent density $\rho(z)$ and wave speed $c(z)$, while the overlying half-space has known ρ_0 and c_0.

The basic acoustic equations in the frequency domain are

$$\rho(z)\,\omega^2 u_x(x,z,\omega)=\partial P/\partial x \quad (1a)$$

$$\rho(z)\,\omega^2 u_Z(x,z,\omega)=\partial P/\partial z \quad (1b)$$

$$P(x,z,\omega)=-\rho(z)c(z)^2(\partial u_x/\partial x+\partial u_z/\partial z) \quad (1c)$$

where u_x and u_z are the horizontal and vertical displacement components of the medium, respectively and P is pressure.

The boundary condition is

$$P(x,z,\omega,\theta)=(e^{-ik_z z}+R(\omega,\theta)\,e^{ik_z z})\,e^{-ik_x x} \quad z\leq 0 \quad (2)$$

and a propagation condition is $z\to\infty$.

Here, $k_x=\omega\sin\theta/c_o=\omega p,\quad k_z=\omega\cos\theta/c_o \quad (3)$

are the horizontal and vertical wavenumbers, respectively, and p is the vertical slowness.

Since no lateral variation of the medium is supposed, $e^{-ik_x x}$ represents the only x-offset dependence. Since, by definition,

$$\cos^2\theta(z)=1-p^2c(z)^2 \quad (4)$$

$$c'(z)=c(z)/\cos\theta(z)=\text{local vertical wave speed} \quad (5)$$

$$\tau(z)=\int_0^z ds/c'(s)=\text{vertical travel time} \quad (6)$$

$$Z(\tau)=\rho(\tau)c'(\tau)=\text{acoustic impedance}, \quad (7)$$

eliminating uz results in

$$\partial P/\partial\tau=\omega^2 Zu_z(\tau,\omega) \quad (8a)$$

$$\partial u_z\partial\tau=-(1/Z)P(\tau,\omega) \quad (8b)$$

The energy-normalized downgoing and upgoing waves (Claerbout, 1976) are

$$\hat{D}(\tau,\omega)=P/Z^{1/2}+i\omega Z^{1/2}u_z \quad (9a)$$

$$\hat{U}(\tau,\omega)=P/Z^{1/2}+i\omega Z^{1/2}u_z \quad (9b)$$

which satisfy the two-component wave system

$$\partial\hat{D}/\partial\tau=-i\omega\hat{D}-r(\tau)\hat{U} \quad (10a)$$

$$\partial\hat{U}/\partial\tau=-r(\tau)D+i\omega\hat{U} \quad (10b)$$

where $r(\tau)=\dfrac{1}{2Z}\dfrac{dZ}{d\tau}=\dfrac{d}{2d\tau}\log Z(\tau) \quad (11)$

Eq. (10) describe a downgoing wave $\hat{D}(\tau,\omega)$ is partially scattered into an upgoing wave $\hat{U}(\tau,\omega)$, and vice-versa. The amount of scattering is determined by the reflectivity function $r(\tau)$.

MODIFIED DOWNWARD CONTINUATION ALGORITHM

The downward continuation algorithm consists of four steps. First initial step is derived from the above equations. Second, the algorithm is modified to compute the density and wave speed at each depth from the least-squares fitting to reflection data at more than two angles of incidence. Third, a threshold is included to eliminate false interfaces due to noisy data. Finally, an additional transmission loss modification is made.

Initial Step

The wave equations (10) lead to a downward continuation algorithm. Taking the inverse Fourier transform of equations (10) result in

$$\frac{\partial D}{\partial\tau}(\tau,t)+\frac{\partial D}{\partial\tau}(\tau,t)=-r(\tau)U\tau,t) \quad (12a)$$

$$\frac{\partial U}{\partial\tau}(\tau,t)+\frac{\partial U}{\partial\tau}(\tau,t)=-r(\tau)D\tau,t) \quad (12b)$$

By causality, $D(\tau,t)$ and $U(\tau,t)$ have the forms

$$D(\tau,t)=\delta(t-\tau)+\tilde{D}(\tau,t)\,1(t-\tau) \quad (13a)$$

$$U(\tau,t)=\tilde{U}(\tau,t)\,1(t-\tau) \quad (13b)$$

where 1( ) is the unit step or Heaviside function which is defined as

$$1(t)=\begin{cases}0, & t<0,\\ 1, & t\geq 0,\end{cases}$$

and δ( ) is the unit impulse or Dirac delta function which is defined as

$$\delta(t)=\begin{cases}0, & t\neq 0,\\ \infty, & t=0.\end{cases}$$

Inserting equations (13) into equations (12) result in

$$\frac{\partial\tilde{D}}{\partial\tau}(\tau,t)+\frac{\partial\tilde{D}}{\partial t}(\tau,t)=-r(\tau)\,\tilde{U}(\tau,t) \quad (14a)$$

$$\frac{\partial\tilde{D}}{\partial\tau}(\tau,t)+\frac{\partial\tilde{U}}{\partial t}(\tau,t)=-r(\tau)\,\tilde{D}(\tau,t) \quad (14a)$$

$$r(\tau)=2\,\tilde{U}(\tau,\tau) \quad (14c)$$

Eq. (14) can be differentially downward continued in τ. By causality both $\tilde{D}(\tau,t)$ and $\tilde{U}(\tau,t)$ are zero for $t<\tau$. The algorithm is initialized by

$$\tilde{D}(0,t)=0,\quad \tilde{U}(0,t)=f^{-1}\{R(\omega\,\theta)\} \tag{15}$$

Once $r(\tau)$ is reconstructed, integration of (11) yields $Z(\tau)$. By running this algorithm, for two different angles of incidence $\theta=\theta_1$, θ_2, it is possible to recover two impedance profiles $Z_1(\tau_1)$ and $Z_2(\tau_2)$. This procedure requires only $O(N^2)$ computations while two Marchenko integral equations require $O(N^3)$ computations. To invert the two impedance profiles $Z_1(\tau_1)$ and $Z_2(\tau_2)$ for the density $\rho(z)$ and wave speed $c(z)$ profiles least-squares fitting at each depth is required.

Least-squares Fitting at Each Depth

From the reflection coefficient $R(\omega,\theta)$ of the medium at different angles of incidence $\theta=\theta_1$, θ_2, $\cdots$, θ_m, where subscript m is the number of incident angles. Let $\theta_i(z)$, $c'_i(z)$ and $\tau_i(z)$ be the quantities defined in Eq. (4)~(6), respectively, associated with incident angle θ_i. Then

$$r_i(z)=\frac{d}{2dz}\log(\rho(z)\,c'_i(z))=\frac{r(\tau_i(z))}{c'_i(z)} \tag{16}$$

The least-square-error solutions to (16) and forward differences to discretize the derivatives of $\rho(z)$ and $c(z)$ yield the following equations

$$\rho(z+\Delta)=\rho(z)+\frac{2\Delta\rho(z)}{TEMP(z)}\Big(\sum_{i=1}^{m}\cos^{-4}\theta_i(z)\sum_{i=1}^{m}r_i(z) - \sum_{i=1}^{m}r_i(z)\,\cos^{-2}\theta_i(z)\sum_{i=1}^{m}\cos^{-2}\theta_i(z) \tag{17a}$$

$$c(z+\Delta)=c(z)+\frac{2\Delta c(z)}{TEMP(z)}\Big(m\sum_{i=1}^{m}r_i(z)\cos^{-2}\theta_i(z) - \sum_{i=1}^{m}\cos^{-2}\theta_i(z)\sum_{i=1}^{m}r_i(z)\Big) \tag{17b}$$

$$TEMP(z)=m\sum_{i=1}^{m}\cos^{-4}\theta_i(z)-\Big(\sum_{i=1}^{m}\cos^{-2}\theta_i(z)\Big)^2 \tag{17c}$$

where Δ is the layer thickness.

After discretization $r(z)$ becomes a series of reflection coefficients. Eq. (17) allow the improved updates to be used as the algorithm runs, thus improving the subsequent updates as well.

Noise Level Threshold

The algorithm interprets $U(\tau,\tau)$ as an interface between two layers of thickness Δ. However, if the medium is actually homogeneous at this depth, $U(\tau,\tau)$ is due to noise, which will be interpreted as a false interface. It is important to distinguish between actual and false interfaces that arise from the noise in the data.

It has been suggested the use of a threshold computed from the condition number of the inverse problem at depth (Bruckstein *et al.*, 1985; Yamamoto and Shon, 1991; Shon and Yamamoto, 1992). It is shown that if $r(z=n\Delta)$ is the reflection coefficient at depth $n\Delta$ and $r(n\Delta)$ is the algorithm-generated reflection coefficient, then

$$|r(z-n\Delta)-r(n\Delta)|\le 2\varepsilon\prod_{i=1}^{n-1}\frac{1+|r(i\Delta)|}{1-|r(i\Delta)|}+O(\varepsilon^2) \tag{18}$$

where ε is the maximum noise level.

Eq. (18) states that deeper layer are more difficult to reconstruct, since the wave is weaker, having been partially scattered at each interface. If the medium consists of thick, homogeneous layers, so that most of the $r(n\Delta)$ are zero, then equation (18) can be used as a threshold: If the computed $r(n\Delta)$ is below the threshold, then it is likely due to noise and is replaced by zero, while if it is above the threshold, then it is likely to be real interface. The bound is computed as the algorithm runs. The advantage of using this threshold is that many false interfaces due to noisy data are eliminated. The disadvantage is that some weak reflections may be eliminated, but these would be lost in the noise anyway (Shon and Yamamoto, 1992).

Transmission Losses

The effects of transmission losses at an interface can easily be incorporated into the algorithm by making a minor modification to the forward-differences approximation to the equations (14). The energy-normalized waves $[D_a, U_a]$ just above an interface are related to the waves $[D_b, U_b]$ just below the interface (Claerbout, 1976) by

$$\begin{bmatrix} U_b \\ D_b \end{bmatrix}=\frac{1}{\sqrt{1-r^2}}\begin{bmatrix} 1 & -r \\ -r & 1 \end{bmatrix}\begin{bmatrix} U_a \\ D_a \end{bmatrix}$$

Transmission losses may be taken into account by dividing the waves $\sqrt{1-r(n\Delta)^2}$ at each recursion.

NUMERICAL RESULTS

The algorithm was tested. In general highly noisy data causes imperfect reconstruction of $c(z)$, resulting in incorrect values of $\tau_i(z)$. This causes the algorithm to examine $U_i(z, \tau_i(z))$ at the wrong time, so that it misses a reflection coefficient. The higher the noise level in the data, the closer to the surface is the brea-

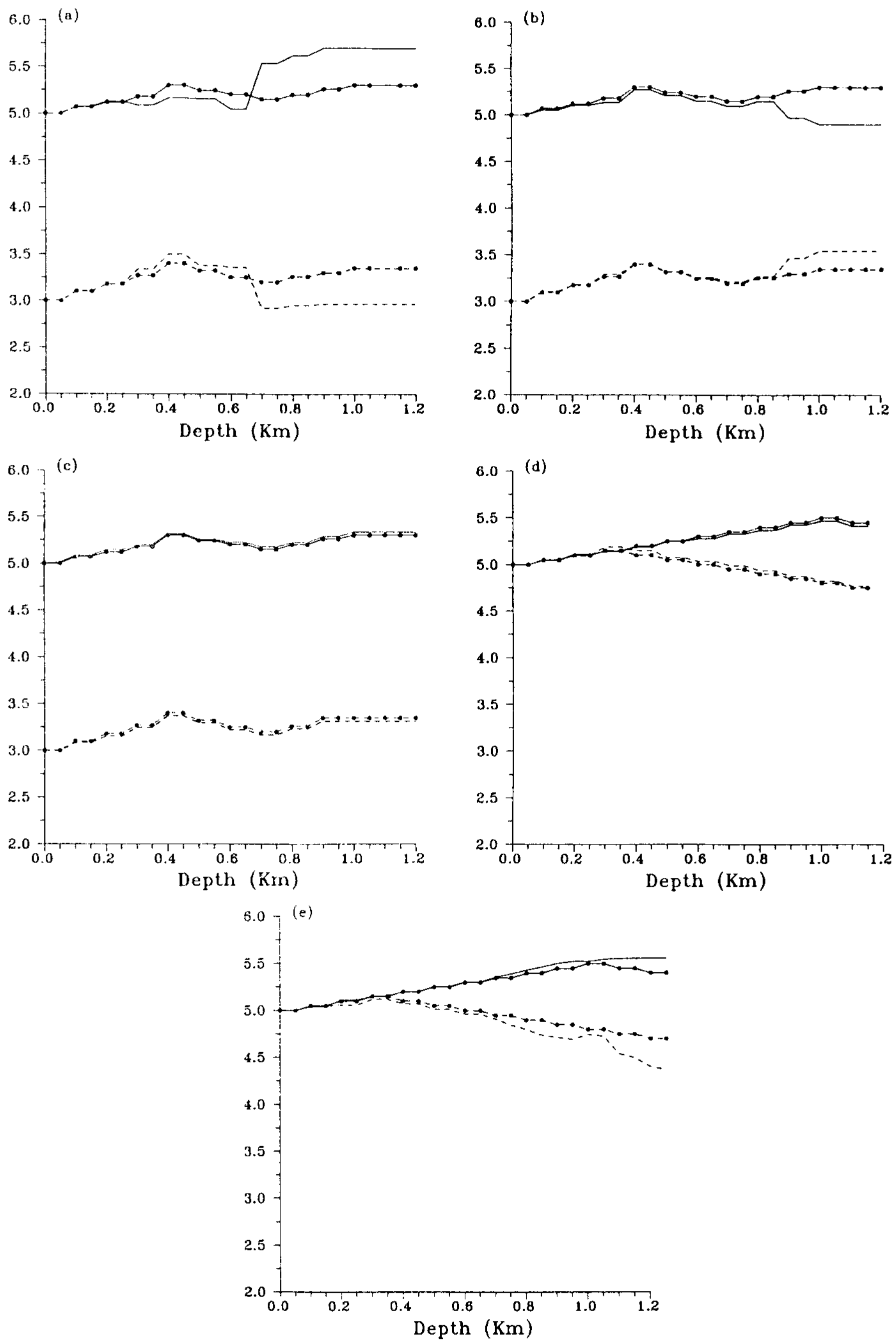

Fig. 1. Plot of the actual (dotted solid-line) and reconstructed (solid-line) profiles of the wave speed in km/sec (upper pair) and the actual (dotted dashed-line) and reconstructed (dashed-line) density profiles relative to water (lower pair) (a) first run, (b) second run, (c) third run, (d) fourth run, and (e) fifth run.

kdown depth. However, the least-squares modification allows a much higher noise level before breakdown occurs at a given depth.

These points are illustrated in Figs. 1. Fig. 1a shows the result of reconstructing a ten-layer medium using reflection data at two angles of incidence. The breakdown occurs at 0.6 km. Fig. 1b shows the result of reconstructing the same medium using reflection data at five angles of incidence, with the same noise level as Fig. 1a. Now the breakdown depth is at 0.85 km. Fig. 1c shows the result of reconstructing the same medium using reflection data at eight angles of incidence, with the same noise level as before. The algorithm is able to reconstruct the medium almost perfectly. Fig. 1d demonstrates the reconstruction of a fifteen-layer medium using data at five angles of incidence, at a fairly high noise level. Fig. 1e demonstrates the reconstruction of a twelve-layer medium using data at ten angles of incidence, at a very high noise level. Despite the high noise level, the reconstruction is fairly accurate through eight layers.

CONCLUSIONS

A downward continuation algorithm for reconstructing the density and wave speed profiles of an acoustic medium from its reflection response at several incident angles has been given. The algorithm requires fewer computations than solving the integral equations by Gaussian elimination. Since the quantities in the algorithm carry such physical interpretations as energy-normalized waves, reflection coefficients, and travel times, it is possible to follow its operation in some detail. The algorithm includes all multiple reflections. Performing a least-squares fitting at each depth requires only relatively minor amounts of additional computation, and allows the improved estimates of $\rho(z)$ and $c(z)$ to be used while the algorithm is running. If the data has a high noise level, the depth at which breakdown occurs can be made deeper by using data at more different angles of incidence.

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하향연속 알고리즘에 의한 반사계수로부터의 속도 및 밀도값 복원

손호웅 · 서만철

요 약: 본 논문에서는 깊이에 따라 밀도값과 (종파)속도값이 변하는 지하 하부층들의 반사계수로부터 밀도와 속도 주상도를 구하는 알고리즘을 개발하고자 하였다. 하향연속(下向連續) 혹은 박층(剝層; layer stripping) 방식 알고리즘은 적분방정식을 이용한 계산법보다 계산 과정을 상당히 줄일 수 있다. 본 논문에서는 여러 방향으로의 입사파에 의한 반사계수를 개선된 하향연속 알고리즘을 이용하여 처리하였으며, 각 깊이에서 최소자승 맞춤(fitting)을 실행하였다. 본 연구결과는 다른 하향연속 알고리즘에 비하여 상당한 성과를 보여주고 있다.