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SMASH PRODUCT ALGEBRAS AND INVARIANT ALGEBRAS

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ABSTRACT. Let H and G be finite dimensional semisimple Hopf algebras and let A and B be left H and G-module algebras respectively. We use smash product algebras to show that 1) if A is right Artinian then A^H is right Artinian, 2) Soc $V_A \subset$ Soc V_{A^H} and rad $V_A \supset$ radV_{A^H}, 3) K dim ${}_B V_A = K$ dim ${}_{BG} V_{A^H}$.

Throughout we let k be a field. Tensor products are assumed to be over k unless stated otherwise. Let H be a Hopf algebra over a field k. We let Δ be the comultiplicatin and we will use the sigma notation, $\Delta\colon H \to H \otimes H$, $\Delta(h) = \Sigma_{(h)} h_1 \otimes h_2$. Let ϵ be the counit and S be the antipode of H.

An algebra A is said to be a *left H-module algebra* if

(1) A is a left H-module, via $h \otimes a \mapsto h \cdot a$
(2) $h \cdot (ab) = \Sigma(h_1 \cdot a)(h_2 \cdot b)$
(3) $h \cdot 1_A = \epsilon(h) 1_A$ for all $h \in H$ and for all $a, b \in A$.

Let A be a left H-module algebra then the *smash product algebra* $A\#H$ is defined as follows: For all $a, b \in A$ and for all $h, k \in H$,

(1) as k-spaces, $A\#H = A \otimes H$. We write $a\#h$ for the element $a \otimes h$.
(2) multiplication is given by

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$$(a\#h)(b\#k) = \Sigma a(h_1 \cdot b)\#h_2 k.$$

We show that $A \cong A\#1$ and $H \cong 1\#H$; for this reason we frequently abbrivate the element $a\#h$ by ah.

REMARK. Let H be a group algebra kG and let A be a H-module algebra. Since $\Delta g = g \otimes g$ for $g \in G$, $g \cdot (ab) = (g \cdot a)(g \cdot b)$ for all $a, b \in A$ and thus g acts as an endomorphism of A. In addition, each g acts as an automorphism of A since $g^{-1}g = 1$. Thus we have a group homomorphism $G \longrightarrow Aut_k A$. Conversely, any such map makes A into a kG-module algebra. In this case $A\#kG = A * G$ is the skew group ring. The multiplication in $A\#kG$ is just $(ag)(bh) = (a\#g)(b\#h) = a(g \cdot b)gh = ab^{g^{-1}}gh$.

We extend some arguments for the skew group rings to finite dimensional Hopf algebras. If H is a finite dimensional Hopf algebra then the left integral of H, $\int_H^l = \{t \in H | ht = \epsilon(h)t$, for all $h \in H\}$, is one dimensional [LS]. Choose $0 \neq t \in \int_H^l$. Let A be a left H-module algebra and let $A^H = \{a \in A | h \cdot a = \epsilon(h)a$ for all $h \in H\}$. Then the map $\hat{t}$: $A \to A$ given by $\hat{t}(a) = t \cdot a$ is an A^H-bimodule map with values in A^H.

LEMMA 1 [CFM]. *Let H be finite dimensional acting on A and assume that $\hat{t}$: $A \to A^H$ is surjective. Then there exists a nonzero idempotent $e \in A\#H$ such that $e(A\#H)e = A^H e \cong A^H$.*

If $\hat{t}$ is surjective , there exists $c \in A$ with $\hat{t}(c) = t \cdot c = 1$. Define $e = tc$ then $e^2 = tctc = (t \cdot c)tc = e$. If H is finite dimensional Hopf algebra then H is semisimple if and only if $\epsilon(\int_H^l) \neq 0$ [LS]. Hence if H is semisimple , we may choose $t \in \int_H^l$ with $\epsilon(t) = 1$. It follows that $\hat{t}(1) = t \cdot 1 = \epsilon(t) \cdot 1 = 1$ and so $\hat{t}(A) = A^H$.

Fix a basis $\{h_1, h_2, \cdots, h_n\}$ of H. Let R be any k-algebra. For any (R, A)-bimodule V, let $W = V \otimes_A (A\#H)$ be the induced $(R, A\#H)$-bimodule. Let $L({}_R V_{A^H})$ denote the lattice of (R, A^H)-subbimodules of V and let $L({}_RW_{A\#H})$ be the lattice of $(R, A\#H)$-subbimodules of W.

LEMMA 2. *Let H be a finite dimensional semisimple Hopf algebra and let R,A,e,V and W be as above. Then there exist inclusion preserving maps*

$$\sigma\colon L({}_RV_{A^H}) \to L({}_RW_{A\#H})$$

and

$$\mu\colon L({}_RW_{A\#H}) \to L({}_RV_{A^H})$$

such that for $U \in L({}_RV_{A^H})$, and $X_1, X_2 \in L({}_RW_{A\#H})$ we have $U^{\sigma\mu} = U$ and $(X_1 \oplus X_2)^\mu = X_1^\mu \oplus X_2^\mu$.

PROOF. Define

$$\sigma\colon L({}_RV_{A^H}) \to L({}_RW_{A\#H}), U \mapsto (U \otimes e)(A\#H)$$

and

$$\mu\colon L({}_RW_{A\#H}) \to L({}_RV_{A^H}), \Sigma v_i \otimes h_i \mapsto \Sigma\epsilon(h_i)v_i,$$

for any $w = \Sigma v_i \otimes h_i \in W$. Then μ is well-defined since any $w \in W$ has a unique representation in this form. μ is an (R, A^H)-bimodule map since $ha = ah$ for all $a \in A^H$. Thus if $X \in L({}_RW_{A\#H})$, then $X^\mu \in L({}_RV_{A^H})$ and if $X_1, X_2 \in L({}_RW_{A\#H})$ then $(X_1 + X_2)^\mu = {X_1}^\mu + {X_2}^\mu$. Clearly both σ and μ preserve inclusions. If $X_1 \cap X_2 = 0$ then $X_1e \cap X_2e = 0$ since $X_1, X_2 \in L({}_RW_{A\#H})$. For any $w = \Sigma v_i \otimes h_i \in W$,

$$\begin{aligned} we &= (\Sigma v_i \otimes h_i)e = \Sigma v_i \otimes h_i e \\ &= \Sigma v_i \otimes h_i tc = \Sigma v_i \otimes \epsilon(h_i)tc \\ &= \Sigma\epsilon(h_i)v_i \otimes tc = \mu(w) \otimes e, \end{aligned}$$

and

$$v \otimes et = (v \otimes e)t$$

$$v \otimes et = v \otimes tct = v \otimes (t \cdot c)t = v \otimes 1t = v \otimes t, \forall v \in V.$$

If $v \otimes e = 0$ then $v \otimes t = v \otimes et = (v \otimes e)t = 0$ so $v = 0$. Thus $v_1 \otimes e = v_2 \otimes e$ implies $v_1 = v_2$. Therefore if $X_1 \cap X_2 = 0$ then $X_1^\mu \cap X_2^\mu = 0$. For then, for any $v \in X_2^\mu \cap X_2^\mu, v = \mu(x_1) = \mu(x_2)$ for some $x_1 \in X_1, x_2 \in X_2$ and $v \otimes e = \mu(x_1) \otimes e = \mu(x_2) \otimes e$. So $v \otimes e = x_1 e = x_2 e \in X_1 e \cap X_2 e = 0$ hence $v = 0$. Therefore $(X_1 \oplus X_2)^\mu = X_1^\mu \oplus X_2^\mu$. For $U \in L({}_R V_{AH}), U^\sigma e = (U \otimes e)(A \# H)e = U \otimes A^H e = U \otimes e$ by Lemma 1. Thus if $w \in U^\sigma, we = \mu(w) \otimes e \in U \otimes e$ and so $\mu(w) \in U$; that is $U^{\sigma\mu} \subseteq U$. If $u \in U$ then $w = u \otimes e = (u \otimes e)(1 \otimes 1) \in U^\sigma$ and $u \otimes e = we = \mu(w) \otimes e$. It follows that $u = \mu(w) \in U^{\sigma\mu}$ and so $U = U^{\sigma\mu}$. This completes the proof.

When H is a finite dimensional semisimple Hopf algebra and A is a left H-module algebra, if A is right Noetherian, then A is right Noetherian A^H-module [M].

Two basic properties of Krull dimension we require are as follows[GW]:

(a) Let R and S be rings, let V and W be modules over R and S respectively and suppose there exists an inclusion preserving one-to-one map $L(V_R) \to L(W_S)$. If $K\dim W_S$ exists then $K\dim V_R$ exists and $K\dim V_R \leqq K\dim W_S$. In particular, this always applies in the special case when $V = W$, $S \subset R$ and where $L(V_R) \to L(V_S)$ is given by restriction of operators to S.

(b) Let V be a submodule of the right R-module W. Then $K\dim W_R$ exists if and only if both $K\dim V_R$ and $K\dim (W/V)_R$ exist and in this case $K\dim W_R = \sup\{K\dim V_R, K\dim (W/V)_R\}$.

LEMMA 3. *If $f: M \to N$ is a group isomorphism for left S-modules M and N for a ring S and M is a right R-module for a ring R then it* *is possible to give a right R-module structure on N and there exists a right R-module isomorphism $\phi\colon M \to N$.*

PROOF. For any $n \in N$, there exists $m \in M$ such that $f(m) = n$. We set $n \cdot r = f(m) \cdot r = f(m \cdot r)$ for $n \in N$, $m \in M$ and $r \in R$. Then N is a right R-module. Define $\phi\colon M \to N$ via $m \mapsto f(m) = \phi(m)$. Then ϕ is a right R-module isomorphism.

PROPOSITION 1. *Let H be a finite dimensional semisimple Hopf algebra and let A be a left H-module algebra. If V is a right A-module then $K\dim V_A$ exists if and only if $K\dim V_{A^H}$ exists and in this case $K\dim V_A = K\dim V_{A^H}$.*

PROOF. Set $W = V \otimes (A\#H)$. By the special case in (a) above, if $K\dim V_{A^H}$ exists then so dose $K\dim V_A$ and, moreover, $K\dim V_A \leqq K\dim V_{A^H}$. Conversely, assume that $K\dim V_A$ exists. Give a right A-module structure of W as $w \cdot a = w(a\#1)$ for $w \in W$ and $a \in A$. Since $A\#H$ is a free left A-module of rank $n = \dim {}_kH$, there exists a left A-module isomorphism $f\colon A\#H \to A^{(n)}$. Give a right A-module structure of $V \otimes_A A^{(n)}$ as $(v \otimes u) \cdot a' = (v \otimes f(ah)) \cdot a' = v \otimes f(ah \cdot a')$ for $v \in V$, $ah \in A\#H$, $u \in V^{(n)}$ and $a' \in A$ and define $\phi\colon V \otimes_A (A\#H) \to V \otimes_A A^{(n)}$ by $id \otimes f$. Then ϕ is a right A-module isomorphism. And $V \otimes_A A^{(n)} \cong (V \otimes_A A)^{(n)} \cong V_A^{(n)}$ as right A-modules by the Lemma 3. Hence (b) implies that $K\dim W_A = K\dim V_A$. Since $A \cong A\#1 \subset A\#H$, $K\dim W_{A\#H}$ exists and in fact $K\dim W_{A\#H} \leqq K\dim W_A = K\dim V_A$ by the special case in (a). We show that Lemma 2 also holds for right A-module V. So Lemma 2 asserts that there exists an one to one inclusion preserving map $\sigma\colon L(V_{A^H}) \to L(W_{A\#H})$. Hence (a) implies that $K\dim V_{A^H} \leqq K\dim W_{A\#H} \leqq K\dim W_A = K\dim V_A$. This completes the proof.

COROLLARY. *A is right Artinian if and only if A is a right Artinian A^H-module.*

THEOREM 1. *If A is right Artinian then A^H is right Artinian.*

PROOF. Assume that A is right Artinian and take $V = A$ in Proposition 1. Since $A^H_{A^H} \subset A_{A^H}$, $K\dim A^H_{A^H} \leqq K \dim A_{A^H} = K\dim A_A$. Therefore A^H is right Artinian.

LEMMA 4. *Let $A\#H$ be a smash product algebra for a finite dimensional semisimple Hopf algebra H. Let W be an $(R, A\#H)$-bimodule for any algebra R and let $V \in L({}_RW_{A\#H})$. If V has a complement in $L({}_RW_A)$, then V also has a complement in $L({}_RW_{A\#H})$.*

PROOF. See [BM 89].

PROPOSITION 2. *Let H be a finite dimensional semisimple Hopf algebra and let A be a left H-module algebra. If V_A is completely reducible then so is V_{A^H}.*

PROOF. If V_A is completely reducible then $W_A = V \otimes (A\#H) \cong V_A^{(n)}$ is also completely reducible. By Lemma 4, we conclude that $W_{A\#H}$ is completely reducible. Therefore if $U \in L(V_{A^H})$ then $U^\sigma \in L(W_{A\#H})$ has a complemet $X \in L(W_{A\#H})$ with $W = U^\sigma \oplus X$. By Lemma 2 , $V = W^\mu = (U^\sigma \oplus X)^\mu = U^{\sigma\mu} \oplus X^\mu = U \oplus X^\mu$. Thus $X^\mu \in L(V_{A^H})$ is a complement for U and we have shown that V_{A^H} is completely reducible.

Recall that the socle of V, Soc V_A , is the sum of all simple submodules of V and the radical of V, rad V_A, is the intersection of all maximal submodules of V.

THEOREM 2. *Soc $V_A \subset$ Soc V_{A^H} and rad $V_A \supset$ rad V_{A^H} where H and A are as above.*

PROOF. By Proposition 2, soc V_A is completely reducible as an A^H-module. Hence soc $V_A \supset$ Soc V_{A^H}. Let M be a maximal A-module of V. Then V/M is a completely reducible A-module. By

Proposition 2, V/M is completely reducible as A^H-module. In particula, $M = \cap L_i$ for certain maximal A^H-submodules of V. It follows that rad $V_A \supset$ rad V_{A^H}.

LEMMA 5. *Let H and G be finite dimensional, semisimple Hopf algebras, A a left H-module algebra and let B be a left G-module algebra. If V is a (B, A)-bimodule then the induced module $W = (B\#G) \otimes_B V \otimes_A (A\#H)$ is a $(B\#G, A\#H)$-bimodule. Furthermore, there exists inclusion preserving maps*

$$\sigma\colon L({}_{B^G}V_{A^H}) \to L({}_{B\#G}W_{A\#H})$$

and

$$\mu\colon L({}_{B\#G}W_{A\#H}) \to L({}_{B^G}V_{A^H})$$

such that for any $U \in L({}_{B^G}V_{A^H}), U^{\sigma\mu} = U$ and μ preserves direct sums.

PROOF. For all $b \in B$ and $g \in G$, $bg = \Sigma g_2(\overline{S}g_1 \cdot b)$; for $\Sigma g_2(\overline{S}g_1 \cdot b) = \Sigma g_2\#(\overline{S}g_1 \cdot b) = \Sigma(1\#g_2)((\overline{S}g_1 \cdot b)\#1) = \Sigma 1 \cdot (g_2 \cdot (\overline{S}g_1 \cdot b)\#g_3 = \Sigma 1 \cdot (g_2\overline{S}g_1 \cdot b)\#g_3 = \Sigma\epsilon(g_1)1_G \cdot b\#g_2 = b\#g = bg$. Define $\beta\colon B\#G \to G \otimes B$ via $bg = \Sigma g_2(\overline{S}g_1 \cdot b) \mapsto \Sigma g_2 \otimes (\overline{S}g_1 \cdot b)$ and $\alpha\colon G \otimes B \to B\#G$ via $g \otimes b \mapsto (1\#g)(b\#1)$. Then β is a right B-module isomorphism with the inverse α. Therefore $B\#G$ is a right B-module. Since G is a finite dimensional Hopf algebra, the antipode S of G is bijective [LS]. Therefore $B\#G$ is a free right B-module with rank $n = \dim_k H$ since S is invertible [BM]. Thus $W = (B\#G) \otimes_B V \otimes_A (A\#H)$ has a proper tensor product structure and it is a $(B\#G, A\#H)$-bimodule. First set $R = B^G$ in Lemma 2 and let $M = V \otimes_A (A\#H)$. Then M is a $(R, A\#H)$-bimodule. By Lemma 2, there exist inclusion preserving maps

$$\sigma_1\colon L({}_{B^G}V_{A^H}) \to L({}_{B^G}M_{A\#H})$$

and

$$\mu_1 \colon L({}_{BG}M_{A\#H}) \to L({}_{BG}V_{AH})$$

such that $\mu_1\sigma_1 = id$ and μ_1 preserves direct sums. We can now apply the left analog of Lemma 2 with $R = A\#H$ for $W = (B\#G) \otimes M$, since $B\#G$ is a free right B-module with rank $n = \dim_k H$. We deduce that there exist inclusion maps

$$\sigma_2 \colon L({}_{BG}M_{A\#H}) \to L({}_{B\#G}M_{A\#H}), U \mapsto (B\#G)(e' \otimes U)$$

and

$$\mu_2 \colon L({}_{B\#G}W_{A\#H}) \to L({}_{BG}M_{A\#H}), \Sigma g_i \otimes m_i \mapsto \Sigma \epsilon(g_i) m_i$$

such that $\sigma_2\mu_2 = id$ and μ_2 preserves dirct sums. It is now clear that $\sigma = \sigma_2\sigma_1$ and $\mu_1\mu_2$ have the appropriate properties.

THEOREM 3. *Let H and G be finite dimensional, semisimple Hopf algebras and let H and G be acting on A and B respectively. If V is a (B, A)-bimodule then $K\dim\, {}_BV_A$ exists if and only if $K\dim\, {}_{BG}V_{AH}$ exists and in this case $K\dim\, {}_BV_A = K\dim\, {}_{BG}V_{AH}$.*

PROOF. Let $W = (B\#G) \otimes_B V \otimes_A (A\#H)$. Since $A\#H$ is a free left A-module of rank $n = \dim_k H$ and $B\#G$ is a free right B-module of rank $m = \dim_k G$ as in the proof of Lemma 5, ${}_BW_A = (B\#G) \otimes_B V \otimes_A (A\#H) \cong (B\#G) \otimes_B V \otimes_A A^{(n)} \cong (B\#G) \otimes_B (V \otimes_A A)^{(n)} \cong (B\#G) \otimes_B V^{(n)} \cong B^{(m)} \otimes_B V^{(n)} \cong (B \otimes_B V^{(n)})^{(m)} \cong (V^{(n)})^{(m)} \cong_B V_A^{(mn)}$, as in the proof of Lemma 4. In view of preceding Lemma, the proof of Proposition 1 immediately can be applicable to yield the result.

COROLLARY. *Assume that H is a finite dimensional,semisimple Hopf algebra and A is a left H-module algebra. If A satisfies the descending chain condition on two sided ideals, then A^H satisfies the descending chain condition on two sided ideals.*

PROOF. Take $A = B$, $H = G$ and $V = A$ in Theorem 3. If A satisfies the descending chain condition on two sided ideals then $K\dim\ {}_{A^H}A^H_{A^H} \leqq K\dim\ {}_{A^H}A_{A^H} = K\dim\ {}_AA_A = 0$. Hence $K\dim$ ${}_{A^H}A^H_{A^H} = 0$. Therefore A^H satisfies the descending chain condition on two sided ideals.

REFERENCES

[BM] R. J. Blattner, S. Montgomery, *Crossed products and Galois extensions of Hopf algebras*, Pacific J. Math. **137** (1989), 37-54.

[CFM] M. Cohen, D. Fischman, S. Montgomery, *Hopf Galois extensions, Smash products, and Morita equivalence*, J.Algebra **133** (1990), 351-372.

[GW] K. R. Goodearl, R. B. Warfield,JR, *An Introduction to Noncommutative Noetherian Rings*, Cambridge University, New York, 1989.

[LS] R. S. Larson, M. Sweedler, *An associative orthogonal bilinea form for Hopf algebras*, Amer. J. Math. **91** (1969), 75-93.

[M] S. Montgomery, *Hopf Algebras and their actions on Rings*, AMS, Providence,Rhode Island, 1993.

[S] M. E. Sweedler, *Hopf Algebras*, Benjamin, New York, 1969.

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