JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 8, June 1995

ONE POINT COMPACTIFICATION IN SEMIFLOWS

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1. Introduction

The purpose of this paper is to show the one point compactifications of dynamical systems. And, we obtain less obvious results i.e. properties of funnel sets. One point compactifications of spaces and dynamical syatems are useful in extending results on compact space to locally compact space.(For example, limit set, nonwandering set so on). Let G denote either the additive topological group R of real number or the additive topological group R^+ of positive real numbers. A *dynamical system* on a topological space X is a continuous map $\Phi : X \times G \to X$ such that, for all $x \in X$ and for all $s, t \in G$, $\Phi(x, s+t) = \Phi(\Phi(x,s),t)$, $\Phi(x,0) = x$. If $G = R$, then the dynamical system Φ is called a *flow* on X, or one parameter group of homeomorphisms of X. If $G = R^+$, then the dynamical system Φ is called a *semiflow* on X. For simplicity, $\Phi(x,t)$ will be denoted by xt. For $A \subset X$ and $S \subset R^+$, let $F(A,S) = \{x \in X : xt \in A \text{ for some } t \in S\}$ and $F(A) = F(A, R^+)$. For $x \in X$, $F(x) = F(\{x\}, R^+)$ is called the *negative funnel* through x.

2. Proof of the theorem

To prove our theorem, we shall use the following lemma:

LEMMA 1. *Let K be compact subset of a T_2 -space X. $F(K,[a,t])$ is a closed set for $0 \le a \le t$.*

Received by the editors on June 30, 1995.

1991 *Mathematics subject classifications*: Primary 58F08.

PROOF. Let $y \in Cl(F(K,[a,t]))$. There exists $(y_i) \in F(K,[a,t])$ such that (y_i) converges to y. For some s_i in $[a,t]$, we have $s_i y_i \in K$. Hence, (s_i) converges to $s \in [a,t]$ and $(y_i s_i)$ converges to $z \in K$, respectively. By the continuity of the semiflow , $(y_i s_i)$ converges to (ys). Also, since X is T_2-space, we obatin $ys = z$. Consequently, $y \in F(K,[a,t])$, as desired.

Let X be a non-compact topological space. There is a standard procedure for associating with X a compact topological space, called the one point compactification X^* of X. Let ∞ denote some point not in X. We define the set X^* to be $X \bigcup \{\infty\}$. To turn X^* into a topological space, we define a subset U of X^* to be open in X^* if and only if either U is an open subset of X or $X^* - U$ is a closed compact subset of X. It follows easily that X^* is indeed compact, and that it is Hausdorff if and only if X is locally compact and Hausdorff. Now let Φ be a dynamacial system on X. We define the one point compactification of Φ to be the map $\Phi^* : X^* \times G \to X^*$ defined by $\Phi^*(x,g) = \Phi(x,g)$ if $x \neq \infty$ and $\Phi(\infty, g) = \infty$.

THEOREM. *Φ^* is a dynamical system on X^*.*

This result was proved by M.C.Irwin(1980) for flows. Remark that we can carry over the results of flows to those of the positive aspects in semiflows. But, studying the negative aspects in semiflows differs from research of flows. Here we give an example which are not true for semiflows, as demonstrated by the following : Let $X = \{(u,v) \in R^2 : u > 0\}$, $G = R^+$, $\Phi((u,v),t) = (u+t, v)$. Then (X, R^+, Φ) is a semiflow which is not extended to X^*.

Now, what we want to show is the following.

THEOREM 2. *Φ^* is a semiflow on X^* if and only if $F(A,K)$ is compact for all compact subsets A of X and for all compact subsets K of R^+.*

PROOF. Necessity. By Lemma1, $F(A,K)$ is closed. Let $\{K_i\}_{i\in I}$ be the family of all compact subsets of R^+. We define an order relation on I by writing $i_1 \le i_2$ if and only if $K_{i_1} \subset K_{i_2}$. Then I is a directed set. The proof is by contradiction. Suppose that our claim were not true. Suppose that for some $i \in I$, $F(A,K) \subset K_i$. This contradicts the fact that $F(A,K)$ is noncompact. Hence for all $i \in I$, $F(A,K)$ is not a subset of K_i. For each $i \in I$, let $x_i \in F(A,K) \bigcap K_i^c$. Since $x_i \in K_i^c$, x_i converges to ∞, and also there exist $t_i \in K$ such that $\Phi(x_i,t_i) \in A$. Since K and A is compact, (t_i) converges to $t \in K$ and $(\Phi(x_i,t_i))$ converges to $a \in A$. Hence $\Phi(x_i,t_i) = \Phi^*(x_i,t_i) \to \Phi^*(\infty,t) = \infty$. This contradicts the fact that the $\infty = a \in A \subset X$, as dersied.

Sufficiency. We claim that $\Phi^* : X^* \times R \to X^*$ is continuous at (∞,t). Let (x_i) converges to $\infty \in X^*$, (t_i) converges to $t \in R^+$. Then $(\Phi(x_i,t_i))$ converges to $x \in X^*$. We want to show that $x = \infty$. Suppose, for contradiction, that $x \neq \infty$. On the other hand, since X is locally compact, there is a neighborhood U of x such that $Cl(U)$ is compact. Also, $\Phi(x_i,t_i) \in U$. Assume that we can $t_i \in [t-1,t+1] = K$. Then $x_i \in F(Cl(U),K)$. Morever, $x_i \to x \in F(Cl(U),K) \subset X$, contradicting the fact that x_i converges to ∞, as desired.

LEMMA 3. *Let U be a neighborhood of $K \subset X$, with both K and $Bd(U)$ compact. Then there exists a neighborhood V of K and a $t > 0$ such that $F(V,[0,t]) \subset U$.*

PROOF. Assume the contrary; Since X is locally compact, there exists a neighborhood W of K with $Cl(W) \subset U$ and $Cl(W)$ compact. Let $(V_i)_{i\in I}$ be the family of all neighborhoods of K contained in W. Choose a sequence (t_n) in R^+ with $t_1 > t_2 > \cdots$ and also, $t_n \to 0$. Let us define a relation on I by setling $i_1 \le i_2$ if and only if $V_{i_1} \subset V_{i_2}$. And also, define a relation on $A = I \times \mathbb{Z}$ by setling $(i_1,n_1) \le (i_2,n_2)$ if and only if $i_1 \le i_2, n_1 \le n_2$. For each (i,n), we have $x_{(i,n)} \in$

$F(V_i, [0, t_n]) - U$. Therefore, $x_{(i,n)}s_{(i,n)} \in V_i \subset W \subset Cl(W)$ for some $s_{(i,n)} \in [0, t_n]$. Since $Cl(W)$ is $x \in Cl(W) \subset U$, we may assume that $x_{(i,n)}s_{(i,n)} \to x \in Cl(W) \subset U$. Now we assert that $x \in K$. If $x \notin K$, there exists a neighborhood G of x and $i_0 \in I$ such that $G \bigcap V_{i_0} = \emptyset$. And also there exists $(i_1, n_1) \in A$ such that if $(i, n) \geq (i_1, n_1)$ then $x_{(i,n)}s_{(i,n)} \in G$. But, if we choose $i_2 \in I$ with $i_2 \geq i_0, i_1$, then we have both $x_{(i_2,n_1)}s_{(i_2,n_1)} \in G$ and $x_{(i_2,n_1)}s_{(i_2,n_1)} \in V_{i_2} \subset V_{i_0}$. This contradicts $V_{i_0} \bigcap G = \emptyset$. Hence we must have $x \in K$. Now since $x_{(i,n)}s_{(i,n)} \to x \in U$, we may assume that $x_{(i,n)}s_{(i,n)} \in U$. In virtue to $x_{(i,n)} \notin U$, it follows that $x_{(i,n)}r_{(i,n)} \in Bd(U)$ for $r_{(i,n)} \in [0, s_{(i,n)}]$. Since $Bd(U)$ is compact, $x_{(i,n)}r_{(i,n)} \to y \in Bd(U)$. However, then

$$x \leftarrow x_{(i,n)}s_{(i,n)} = (x_{(i,n)}r_{(i,n)})(s_{(r,n)} - r_{(r,n)}) \to y0 = y$$

and $x = y$ (X being Hausdorff). This contradicts the fact that $x \in Int(U)$ and $y \in Bd(U)$.

THEOREM 4. *For every x in a locally compact phase space X and sufficiently small $t > 0$, the sets $F(x, [0, t])$ and $F(x, t)$ are compact.*

PROOF. Take a comapct neighborhood U of x, then according to Lemma 3, $F(x, t) \subset F(x, [0, t]) \subset U$ for small $t > 0$. And, both the sets are closed. Since U is compact, these sets are also compact.

ACKNOWLEDGEMENT - We are grateful to Prof. Chin Ku Chu and Prof. Jong Suh Park for fruitful discussions. This work was supported in part by a grant from Hoseo University (1995.4-1996.3).

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