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PETTIS INTEGRABILITY

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ABSTRACT. Let (Ω, Σ, μ) be a finite perfect measure space, and let $f : \Omega \to X$ be strongly measurable. f is Pettis integrable if and only if there is a sequence (f_n) of Pettis integrable functions from Ω into X such that

(a) there is a positive increasing function ϕ defined on $[0, \infty)$ such that $\lim_{t\to\infty} \frac{\phi(t)}{t} = \infty$ and $\sup \int_\Omega \phi(|x^* f_n|)\, d\mu < \infty$ for each $x^* \in B_{X^*}, n \in N$, and
(b) for each $x^* \in X^*$, $\lim_{n\to\infty} x^* f_n = x^* f$a.e..

1. Preliminaries

Let (Ω, Σ, μ) be a finite measure space and let X be a Banach space. The dual of a Banach space X will be denoted by X^* and its closed unit ball will be denoted by B_{X^*}.

A function f from Ω into X is *weakly measurable* if the scalar function $x^* f$ is measurable for each x^* in the dual space X^*.

A function f from Ω into X is said to be *Pettis integrable* if

(a) $x^* f$ is measurable for all $x^* \in X^*$,
(b) $x^* f \in L^1(\mu)$ for all $x^* \in X^*$, and
(c) for each $E \in \Sigma$ there exists an element $\int_E f\, d\mu \in X$ such that

$$< \int_E f\, d\mu, x^* > = \int_E x^* f\, d\mu \quad \text{for all} x^* \in X^*.$$

A function $f : \Omega \to X$ is said to be *(storongly)measurable* if there exists a sequence $(f_n)_{n\in N}$ of μ-simple functions which converges μ-a.e. to f.

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A finite measure space (Ω, Σ, μ) is *perfect* if for each measurable $\psi : \Omega \to R$ and for each set $E \subset R$ such that $\psi^{-1}(E) \in \Sigma$, there is a Borel set $B \subset E$ such that $\mu[\psi^{-1}(B)] = \mu[\psi^{-1}(E)]$.

The class of perfect measure space is very broad. In particular, all Radon measure spaces are perfect.

We shall denote by $\mathcal{P}(\mu, X)$ the space of(class of) Pettis integrable functions $f : \Omega \to X$, endowed with its natural norm given by the formula

$$\|f\| = sup\{\int_\Omega |x^* f| \, d\mu : x^* \in B_{X^*}\}.$$

Note that $\| \int_A f \, d\mu\| \leq \|f\|$ for $f \in \mathcal{P}(\mu, X)$ and $A \in \Sigma$.

2. Pettis Integrability

We are going to need some fact about Pettis integrability. The following proposition can be found in [3] and [6].

PROPOSITION 1[6, Theorem 2.10]. *Let (Ω, Σ, μ) be a finite perfect measure space and let $f : \Omega \to X$. If there is a sequence (f_n) of Pettis integrable functions from Ω into X such that*

(a) *the set $\{x^* f_n : x^* \in B_{X^*}, n = 1, 2, \dots\}$ is uniformly integrable, and*

(b) *for each $x^* \in X^*$, $\lim_{n\to\infty} x^* f_n = x^* f$a.e., then f is Pettis integrable and $\lim_{n\to\infty} \int f_n \, d\mu = \int_E f \, d\mu$ weakly for each $E \in \Sigma$.*

PROOF. If a function $f : \Omega \to X$ is the almost everywhere weak pointwise limit of a sequence (f_n) of Pettis integrable functions in the sense that for each $x^* \in X^*$, $x^* f = \lim_{n\to\infty} x^* f_n$a.e., then f is determined by a WCG subspace of X, and since $\{x^* f_n : x^* \in B_{X^*}, n = 1, 2, \dots\}$ is uniformly integrable, f is Dunford integrable with countably additive indefinite integral. Therefore f is Pettis integrable.

COROLLARY 2[6, Corollary 2.11]. *Let $f : \Omega \to X$ be Dunford integrable, and assume X has no copy of C_0. The following statements are equivalent:*

(a) *f is Pettis integrable.*

(b) *There exists a sequence (f_n) of Pettis integrable functions from Ω into X such that for each $x^* \in X^*$, $\lim_{n\to\infty} x^* f_n = x^* f$a.e..*

PROPOSITION 3. *If (f_n) is a sequence of Pettis integrable functions converging weakly in measure to f, and if $\lim_{n\to\infty} \int_E f_n \, d\mu$ exists for every $E \in \Sigma$, then f is Pettis integrable and $x_E \equiv \lim_{n\to\infty} \int_E f_n \, d\mu = \int_E f \, d\mu$.*

PROOF. Since $x^* f_n$ converges to $x^* f$ in measure and since also $x^* x_E = \lim_{n\to\infty} x^*(\int_E f_n \, d\mu) = \lim_{n\to\infty} \int_E x^* f_n \, d\mu$, from real-function theory it follows that $x^* f$ is integrable and that

$$x^*(x_E) = \lim_{n\to\infty} \int_E x^* f_n \, d\mu = \int_E x^* f \, d\mu$$

for all $x^* \in X^*$ and any $E \in \Sigma$. Thus f is Pettis integrable and $\int_E f \, d\mu = x_E = \lim_{n\to\infty} \int_E f_n \, d\mu$.

THEOREM 4. *Let (Ω, Σ, μ) be a finite perfect measure space and let $f : \Omega \to X$. If there is a sequence (f_n) of Pettis integrable functions from Ω into X such that*

(a) *$x^* f \in L^1$, and*

(b) *for each $x^* \in B_{X^*}$, $\lim_{n\to\infty} x^* f_n = x^* f$ in L^1-norm,*

then f is Pettis integrable and $\lim_{n\to\infty} \int_E f_n \, d\mu = \int_E f \, d\mu$ weakly for each $E \in \Sigma$.

PROOF. If for each x^* in X^* $\lim_{n\to\infty} x^* f_n = x^* f$ in L^1-norm, then $\lim_{n\to\infty} x^* f_n = x^* f$ in measure. If $E \in \Sigma$,

$$\int_E |x^* f_n| \, d\mu \le \int_E |x^* f| \, d\mu + \|x^* f_n - x^* f\|_1 \quad \text{for all} n \ge 1,$$

so $\sup_n \int_\Omega |x^* f_n|\, d\mu < \infty$ in particular. Given $\varepsilon > 0$, take n_0 such that $\|x^* f_n - x^* f\|_1 < \frac{\varepsilon}{2}$ for $n \geq n_0$. Now consider the finite sequence $\mathcal{F} = \{x^* f_1, x^* f_2, \dots, x^* f_{n_0}, x^* f\}$. This is uniformly integrable. Hence there is a $\delta > 0$ such that $\int_E |g|\, d\mu < \frac{\varepsilon}{2}$ whenever $g \in \mathcal{F}$ and $\mu(E) < \delta$. So $\int_E |x^* f_n|\, d\mu < \varepsilon$ for all $n \geq 1$ if $\mu(E) < \delta$. Hence $\{x^* f_n : x^* \in B_{X^*}, n \in N\}$ is uniformly integrable. Therefore by a theorem of Geitz[3, Theorem 3], f is Pettis integrable and $\lim_{n\to\infty} \int_E f_n\, d\mu = \int_E f\, d\mu$ weakly for each $E \in \Sigma$.

Using the above Theorem with [3, Theorem 3] we offer the following:

COROLLARY 5. *Let (Ω, Σ, μ) be a finite perfect measure space, and let $f : \Omega \to X$. Then f is Pettis integrable if and only if there is a sequence (f_n) of simple functions from Ω into X such that*

(a) *the set $\{x^* f_n : x^* \in B_{X^*}, n \in N\}$ is bounded by some element in $L^1(\mu)$, and*

(b) *for each x^* in X^* $\lim_{n\to\infty} x^* f_n = x^* f$ a.e..*

In the following theorem, we replace the condition (a) of [3. Theorem 6] by the existence of some scalar function which guarantee the uniform integrability of the condition.

THEOREM 6. *Let (Ω, Σ, μ) be a finite perfect measure space, and let $f : \Omega \to X$ be strongly measurable, f is Pettis integrable if and only if there is a sequence (f_n) of Pettis integrable functions from Ω into X such that*

(a) *there is a positive increasing function ϕ defined on $[0, \infty)$ such that $\lim_{t\to\infty} \frac{\phi(t)}{t} = +\infty$ and $\sup\{\int_\Omega \phi(|x^* f_n|)\, d\mu : x^* \in B_{X^*}, n \in N\} < \infty$, and*

(b) *for each $x^* \in X^*, \lim_{n\to\infty} x^* f_n = x^* f$ a.e.*

PROOF. ($\Longleftarrow$). Let $M = \sup \int_\Omega \phi(|x^* f_n|)\, d\mu$ and suppose $\varepsilon > 0$

is given. Put $a = \frac{M}{\varepsilon}$ and then choose t_0 such that $\frac{\phi(t)}{t} \geq a$ for $t > t_0$. Hence on the set $\{|x^* f_n| \geq t_0\}$ we have

$$|x^* f_n| \leq \frac{\phi(|x^* f_n|)}{a}.$$

So

$$\begin{aligned}\int_{\{|x^* f_n| \geq t_0\}} |f|\, d\mu &\leq \frac{1}{a} \int_{\{|x^* f_n| \geq t_0\}} \phi(|x^* f_n|)\, d\mu \\ &\leq \frac{M}{a} \\ &= \varepsilon\end{aligned}$$

for all $x^* \in B_{X^*}, n \in N$. We can find t_0 for any given $\varepsilon > 0$. Hence by definition of uniformly integrable, $\{x^* f_n : x^* \in B_{X^*}, n \in N\}$ is uniformly integrable. Then by a theorem of Geitz[3, Theorem 6], f is Pettis integrable.

($\Longrightarrow$). Suppose f is Pettis integrable. By a theorem of Pettis[1, Theorem 8], $\lim_{\mu(E) \to 0} \int_E |x^* f|\, d\mu = 0$ uniformly for $x^* \in B_{X^*}$. Also $\sup_{x^*, f_n} \|x^* f_n\|_{L^1} < \infty$. An appeal to Lavallee Poussin's Theorem[5], establishes the existence of a positive increasing function ϕ defined on $[0, \infty)$ such that $\lim_{t \to \infty} \frac{\phi(t)}{t} = \infty$ such that

$$\sup_{x^*, f_n} \int_\Omega \phi(|x^* f_n|)\, d\mu < \infty$$

and by a theorem of Geitz[3, Theorem 6], for each $x^* \in X^*$, $\lim_{n \to \infty} x^* f_n = x^* f$a.e..

THEOREM 7. *Let (Ω, Σ, μ) be a finite measure space, $f, (f_n)_{n \in N} \subset \mathcal{P}(\mu, X)$ Pettis integrable function $\Omega \to X$, and f is bounded. Then the sequence $(f_n)_{n \in N}$ converges weakly to f with respect to the*

Pettis topology on $\mathcal{P}(\mu, X)$ if and only if $(f_n)_{n\in N}$ is bounded and $(\int_E x^* f_n \, d\mu)_{n\in N}$ converges to $\int_E x^* f \, d\mu$ for all $E \in \Sigma$ and $x^* \in B_{X^*}$.

PROOF. ($\Longrightarrow$). Since the sequence $(f_n)_{n\in N}$ converges weakly to f to Pettis's norm topology, then

$$\|f_n - f\| = \sup\{\int_\Omega |x^*(f_n - f)| \, d\mu : x^* \in B_{X^*}\} \to 0$$

as $n \to \infty$ and so

$$\| \int_E (f_n - f) \, d\mu\| \to 0 \quad \text{as} n \to \infty$$

since $f, (f_n), n \in N$ are Pettis integrable. Hence $(\int_E x^* f_n \, d\mu)_{n\in N}$ converges to $\int_E x^* f$ for all $E \in \Sigma$ and $x^* \in B_{X^*}$, and since f is bound, $(f_n)_{n\in N}$ is bounded.

($\Longleftarrow$). It's clear.

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