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RECURRENT POINTS OF THE CIRCLE MAP

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ABSTRACT. In this paper, we study the inclusion realtion between recursive sets. And we prove that if $\overline{R(f)} \setminus R(f)$ is not empty, then it is infinite, and we characterize the necessary and sufficent condition for which $\overline{R(f)} \setminus R(f)$ is countable.

1. Introduction

Let I be the unit interval, S^1 the circle and X be a topological space. And let $C^0(X,X)$ denote the set of continuous maps from X into itself.

Let $f \in C^0(X,X)$. For any positive integer n, we define f^n inductively by $f^1 = f$ and $f^{n+1} = f \circ f^n$. Let f^0 denote the identity map of X.

For any $f \in C^0(X,X)$, let $P(f)$, $R(f)$, $\Lambda(f)$, $\Gamma(f)$ and $\Omega(f)$ denote the set of periodic points, recurrent points, ω-limit points, γ-limit points and nonwandering points of f, respectively.

Let Y be a subset of I. $\overline{Y}_+$ denotes the right-side closure, and $\overline{Y}_-$ denotes the left-side closure of Y. For any $f \in C^0(I,I)$, J. C. Xiong [4] proved that $(\overline{P(f)}_+ \cap \overline{P(f)}_-) \subset \Gamma(f)$. And also characterized [3] the necessary and sufficient condition for which $\overline{R(f)} \setminus R(f)$ is countable, and proved that if $\overline{R(f)} \setminus R(f)$ is not empty, then it is infinite.

In this paper, we obtain the following similar results for maps of the circle:

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THEOREM A. *Let* $f \in C^0(S^1, S^1)$. *Then we have*

$$(\overline{R(f)_+} \cap \overline{R(f)_-}) \subset \Gamma(f).$$

THEOREM B. *Let* $f \in C^0(S^1, S^1)$. *If* $\overline{R(f)} \setminus R(f)$ *is not empty, then it is infinite.*

THEOREM C. *Let* $f \in C^0(S^1, S^1)$. *Then the followings are equivalent*

(1) $\overline{R(f)} \setminus R(f)$ *is countable.*

(2) $\Gamma(f) \setminus R(f)$ *is countable.*

(3) $(\overline{R(f)_+} \cap \overline{R(f)_-}) \setminus R(f)$ *is countable.*

2. Preliminaries and definitions

Let (X, d) be a metric space and $f \in C^0(X, X)$. A point $x \in X$ is called a *periodic point* of f if for some positive integer n, $f^n(x) = x$. The period of x is the least such integer n. We denote the set of periodic points of f by $P(f)$.

A point $x \in X$ is called a *recurrent point* of f if there exists a sequence $\{n_i\}$ of positive integers with $n_i \to \infty$ such that $f^{n_i}(x) \to x$. We denote the set of recurrent points of f by $R(f)$.

A point $x \in X$ is called a *nonwandering point* of f if for every neighborhood U of x, there exists a positive integer m such that $f^m(U) \cap U \neq \phi$. We denote the set of nonwandering points of f by $\Omega(f)$.

A point $y \in X$ is called an *ω-limit point* of x if there exists a sequence $\{n_i\}$ of positive integers with $n_i \to \infty$ such that $f^{n_i}(x) \to y$. We denote the set of ω-limit points of x by $\omega(x)$. Define $\Lambda(f) = \bigcup_{x \in X} \omega(x)$.

A point $y \in X$ is called an *α-limit point* of x if there exist a sequence $\{n_i\}$ of positive integers with $n_i \to \infty$ and a sequence $\{y_i\}$

of points such that $f^{n_i}(y_i) = x$ and $y_i \to y$. The symbol $\alpha(x)$ denotes the set of α-limit points of x.

A point $y \in X$ is called a *γ-limit point* of x if $y \in \omega(x) \cap \alpha(x)$. The symbol $\gamma(x)$ denotes the set of γ-limit points of x and $\Gamma(f) = \bigcup_{x \in X} \gamma(x)$.

We will use the symbols $\omega_+(x)$ (*resp.* $\omega_-(x)$) to denote the set of all points $y \in X$ such that there exists a sequence $\{n_i\}$ of positive integers with $n_i \to \infty$ such that $f^{n_i}(x) \to y$ and $y < \cdots < f^{n_i}(x) < \cdots < f^{n_2}(x) < f^{n_1}(x)$ (*resp.* $f^{n_1}(x) < f^{n_2}(x) < \cdots < f^{n_i}(x) < \cdots < y$). It is clear that if $x \notin P(f)$, then $\omega(x) = \omega_+(x) \cup \omega_-(x)$. Define $\Lambda_+(f) = \bigcup_{x \in X} \omega_+(x)$ and $\Lambda_-(f) = \bigcup_{x \in X} \omega_-(x)$.

Let Y be an arc in S^1, and let $\overline{Y}$ denote the closure of Y as usual. A point $y \in X$ is called a *right-sided* (resp. *left-sided*) *accumulation point* of Y if for any $z \in X$, $(y, z) \cap Y \neq \phi$ (*resp.* $(z, y) \cap Y \neq \phi$).

The right-side closure $\overline{Y}_+$ (*resp.* left-side closure $\overline{Y}_-$) is the union of Y and the set of right-sided (*resp.* left-sided) accumulation points of Y. A point which is both a right-sided and a left-sided accumulation point of Y is called a *two-sided accumulation point* of Y.

The *forward orbit* $Orb(x)$ of $x \in X$ is the set $\{f^k(x) \mid k = 0, 1, 2, \cdots \}$. Usually the forward orbit of x is simply called the *orbit* of x.

3. Main Results

The idea of the proof of the following lemma is due to [3].

LEMMA 1. *Let $f \in C^0(S^1, S^1)$. Then the set $(\overline{R(f)}_+ \setminus \overline{R(f)}_-) \cup (\overline{R(f)}_- \setminus \overline{R(f)}_+)$ is countable.*

PROOF. For each $y \in (\overline{R(f)}_+ \setminus \overline{R(f)}_-)$, there exists $v_y \in S^1$ such that $(v_y, y) \cap R(f) = \phi$. The family $\{(v_y, y) | y \in \overline{R(f)}_+ \setminus \overline{R(f)}_-\}$ is countable because it is disjoint. Hence $\overline{R(f)}_+ \setminus \overline{R(f)}_-$ is countable. Similarly, $\overline{R(f)_-} \setminus \overline{R(f)_+}$ is also countable. Therefore $(\overline{R(f)_+} \setminus \overline{R(f)_-}) \cup (\overline{R(f)_-} \setminus \overline{R(f)_+})$ is countable.

The following proposition found in [2]

PRPOPOSITION. *Let $f \in C^0(S^1, S^1)$. Then we have*

$$P(f) \subset R(f) \subset \Gamma(f) \subset \overline{R(f)} \subset \Lambda(f) \subset \Omega(f).$$

The following lemma found in [2]

LEMMA 2. *Let $f \in C^0(S^1, S^1)$ and $I = [a, b]$ be an arc for some $a, b \in S^1$ with $a \neq b$, and let $I \cap P(f) = \phi$.*

(a) Suppose that there exists $x \in I$ such that $f(x) \in I$ and $x < f(x)$. Then

(1) if $y \in I, f(y) \in I$, $x < y$ and $f(y) < y$, then $[x, y]$ f-covers $[f(x), b]$, and

(2) if $y \in I, f(y) \notin I$ and

(i) $y < x$, then $[y, x]$ f-covers $[f(x), f(y)]$.

(ii) $x < y$, then $[x, y]$ f-covers $[f(x), f(y)]$.

(b) Suppose that there exists $x \in I$ such that $f(x) \in I$ and $x > f(x)$. Then

(1) if $y \in I, f(y) \in I$, $y < x$ and $y < f(y)$, then $[x, y]$ f-covers $[a, f(x)]$, and

(2) if $y \in I, f(y) \notin I$ and

(i) $y < x$, then $[y, x]$ f-covers $[f(y), f(x)]$.

(ii) $x < y$, then $[x, y]$ f-covers $[f(y), f(x)]$.

The following lemma found in [5]

LEMMA 3. *Let $f \in C^0(S^1, S^1)$. Then we have*

(1) $\overline{R(f)_+} \setminus R(f) \subset \Lambda_+(f)$.

(2) $\overline{R(f)_-} \setminus R(f) \subset \Lambda_-(f)$.

THEOREM A. *Let* $f \in C^0(S^1, S^1)$. *Then we have* $(\overline{R(f)_+} \cap \overline{R(f)_-}) \subset \Gamma(f)$.

PROOF. If $P(f) = \phi$, then we have drsired result since $\overline{R(f)} = \Gamma(f)$ [5]. Suppose $P(f) \neq \phi$. If $z \in R(f)$, then obviousely $z \in \Gamma(f)$. Let $z \in (\overline{R(f)_+} \cap \overline{R(f)_-}) \setminus R(f)$. Then there exists $a, b \in S^1$ with $a < b$ such that $z \in (a, b)$ and $(a, b) \cap Orb(z) = \phi$. By Leamma 3, $z \in \Lambda_+(f) \cap \Lambda_-(f)$. Hence there exist $y_1, y_2 \in S^1$ such that $a < y_1 < z < y_2 < b$ with $z \in \omega(y_1) \cap \omega(y_2)$. Since $\overline{P(f)} = \overline{R(f)}$, $z \in (\overline{P(f)_+} \cap \overline{P(f)_-}) \setminus P(f)$. There exists u_i of periodic point of f with $a < y_1 < u_1 < u_2 < \cdots < z$ and $u_i \to z$. Let p_i be the period of u_i with respect to f. Then $f^{p_i}(u_i) = u_i$ for all$i \geq 1$. The either

$$[u_i, z]\ f^{p_i} - \text{covers } [a, u_i]$$

or

$$[u_i, z]\ f^{p_i} - \text{covers } [u_i, b].$$

We may assume that for infinitely many i, either

$$[u_i, z]\ f^{p_i} - \text{covers } [a, u_i]$$

or

$$[u_i, z]\ f^{p_i} - \text{covers } [u_i, b].$$

Then we consider two cases.

Case I. $[u_i, z]\ f^{p_i}$ – covers $[a, u_i]$ for infinitly many i.

There exists $z_i \in [u_i, z]$ such that $f^{p_i}(z_i) = y_1$. Since $u_i \to z, z_i \to z$. Thus $z \in \alpha(y_1)$, and hence $z \in \omega(y_1) \cap \alpha(y_1) \subset \Gamma(f)$.

Case II. $[u_i, z]\ f^{p_i}$ – covers $[u_i, b]$ for infinitly many i.

There exists $z_i' \in [u_i, z]$ such that $f^{p_i}(z_i') = y_2$. Since $u_i \to z, z_i' \to z$. Thus $z \in \alpha(y_2)$, and hence $z \in \omega(y_2) \cap \alpha(y_2) \subset \Gamma(f)$. The proof of theorem is completed.

THEOREM B. *Let* $f \in C^0(S^1, S^1)$. *If* $\overline{R(f)} \setminus R(f)$ *is not empty, then it is infinite.*

PROOF. It is well known that $f(\overline{R(f)}) = \overline{R(f)}$. Suppose that $\overline{R(f)} \setminus R(f) \neq \phi$. Let $x \in \overline{R(f)} \setminus R(f)$. Inductively, we can choose a sequence of points $x_1, x_2, x_3, \cdots \in \overline{R(f)}$ such that $f(x_n) = x_{n-1}$ for all $n \geq 1$, where $x_0 = x$. Note that $x_n \in R(f)$ for some $n > 0$ implies $x = f^n(x_n) \in R(f)$ and that $x_n = x_m$ for some $m, n > 0$, with $m \neq n$ implies $x \in P(f) \subset R(f)$. Hence $x_1, x_2, x_3, \cdots \in \overline{R(f)} \setminus R(f)$ are different each another. Thus $\overline{R(f)} \setminus R(f)$, which contains an infinitely countable subset $\{x_1, x_2, x_3, \cdots\}$, is inifnite.

THEOREM C. *Let* $f \in C^0(S^1, S^1)$. *Then the followings are equivalent*

(1) $\overline{R(f)} \setminus R(f)$ *is countable.*
(2) $\Gamma(f) \setminus R(f)$ *is countable.*
(3) $(\overline{R(f)}_+ \cap \overline{R(f)}_-) \setminus R(f)$ *is countable.*

PROOF. (1) $\Rightarrow$ (2): Obvious by Proposition.
(2) $\Rightarrow$ (3): Obvious by Thoerem A.
(3) $\Rightarrow$ (1):

$$\begin{aligned}\overline{R(f)} \setminus R(f) &= [\overline{R(f)} \setminus (\overline{R(f)}_+ \cap \overline{R(f)}_-)] \cup [(\overline{R(f)}_+ \cap \overline{R(f)}_-) \setminus R(f)] \\ &= [(\overline{R(f)}_+ \setminus \overline{R(f)}_-) \cup (\overline{R(f)}_- \setminus \overline{R(f)}_+)] \\ &\quad \cup [(\overline{R(f)}_+ \cap \overline{R(f)}_-) \setminus R(f)]\end{aligned}$$

is countable by the condition (3) and Lemma 1. The proof is completed.

COROLLARY. *Let* $f \in C^0(S^1, S^1)$. *If* $(\overline{R(f)}_+ \cap \overline{R(f)}_-) \setminus R(f) = \phi$. *Then* $\overline{R(f)} \setminus R(f)$ *is countable.*

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