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ON THE MCSHANE AND HENSTOCK INTEGRALS OF BANACH–VALUED FUNCTIONS

JAE MYUNG PARK, *DEOK HO LEE AND SEUNG HOON BANG

ABSTRACT. In this paper, we study the properties of the McShane and Henstock integrals for the case in which the function has values in a Banach space.

The McShane and Henstock integrals have been studied for real-valued functions. The McShane integral coincides with the ordinary Lebesgue integral, but the Henstock integral is a proper extension of the Lebesgue integral for real valued functions ([9], S 8.2 and 3.2).

In this paper, we develop the properties of these integrals for the case in which the function has values in a Banach space.

Throughout this paper X will denote a real Banach space and X^* its dual.

DEFINITION 1. We recall the following definitions. Let (S, Σ, μ) be a probability space.

(a) A function $f : S \longrightarrow X$ is *Bochner integrable*, with *Bochner integral* w, if for every $\varepsilon > 0$ we can find a partition $E_0, \dots, E_n$ of S into measurable sets and vectors $x_0, \dots, x_n \in X$ and an integrable function $h : S \longrightarrow R$ such that

$$\int h \leq \varepsilon, \quad ||\phi(t) - x_i|| \leq h(t)$$

for $t \in E_i$, $i \leq n$ and $||w - \sum_{i \leq n} \mu E_i x_i|| \leq \varepsilon$.

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(b) A function $f : S \longrightarrow X$ is *Pettis integrable* if for every $E \in \Sigma$ there is an $x_E \in X$ such that $\int_E x^* f d\mu$ exists and is equal to $x^*(x_E)$ for every $x^* \in X^*$; in this case x_E is the *Pettis integral* of f, and the map $E \longrightarrow x_E : \Sigma \longrightarrow X$ is the *indefinite Pettis integral* of f.

(c) A *McShane partition* of $[0,1]$ is a finite sequence $\langle [a_i, b_i], t_i \rangle_{i \leq n}$ such that $\langle [a_i, b_i] \rangle_{i \leq n}$ is a non-overlapping family of intervals covering $[0,1]$ and $t_i \in [0,1]$ for each i. A *gauge* on $[0,1]$ is a function $\delta : [0,1] \longrightarrow (0, \infty)$. A McShane partition $\langle [a_i, b_i], t_i \rangle_{i \leq n}$ is a *subordinate* to a gauge δ if $t_i - \delta(t_i) \leq a_i \leq b_i \leq t_i + \delta(t_i)$ for every $i \leq n$. We say that a function $f : [0,1] \longrightarrow X$ is *McShane integrable*, with *McShane integral* w, if for every $\varepsilon > 0$ there is a gauge $\delta : [0,1] \longrightarrow (0, \infty)$ such that

$$||w - \sum_{i \leq n} (b_i - a_i) f(t_i)|| \leq \varepsilon$$

for every McShane partition $\langle [a_i, b_i], t_i \rangle_{i \leq n}$ of $[0,1]$ subordinate to δ.

(d) A *Henstock partition* of $[0,1]$ is a McShane partition $\langle [a_i, b_i], t_i \rangle_{i \leq n}$ of $[0,1]$ such that $t_i \in [a_i, b_i]$ for every $i \leq n$. A function $f : [0,1] \longrightarrow X$ is *Henstock integrable*, with *Henstock integral* w, if for every $\varepsilon > 0$ there is a gauge $\delta : [0,1] \longrightarrow (0, \infty)$ such that

$$||w - \sum_{i \leq n} (b_i - a_i) f(t_i)|| \leq \varepsilon$$

for every Henstock partition $\langle [a_i, b_i], t_i \rangle_{i \leq n}$ of $[0,1]$ subordinate to δ.

The next example shows that a McShane integrable function need not measurable.

EXAMPLE 2. The function

$$f : [0,1] \longrightarrow L_\infty[0,1]$$

defined by $f(t) = \chi_{[0,t]}$ is McShane integrable, with McShane integral v, where $v(t) = 1 - t$ for all $t \in [0,1]$. But it is clear that f is not measurable, since f is not essentially separably valued.

Since the McShane integral coincides with the ordinary Lebesgue integral for the real-valued functions, every McShane integrable real-valued function is measurable.

Let $F : [0,1] \longrightarrow X$ and let $t \in (0,1)$. Then we say that F is *approximately differentiable a.e.* on $[0,1]$ if there exists a measurable set $E \subset [0.1]$ that has t as a point of density such that

$$\lim_{\substack{s \to t \\ s \in E}} \frac{F(s) - F(t)}{s - t} = z \quad \text{exists in } X.$$

We will write $F'_{ap}(t) = z$.

THEOREM 3. *Let $f : [0,1] \longrightarrow X$ be McShane integrable on $[0,1]$ and let $F(t) = \int_a^t f$. If F is approximately differentiable a.e. on $[0,1]$ and $F'_{ap} = f$ a.e. on $[0,1]$, then f is measurable.*

PROOF. Since x^*f is McShane integrable for each $x^* \in X^*$, it follows that x^*f is measurable, so that f is weakly measurable.

Since F is continuous ([7], Theorem 8), the set $\{F(t) : t \in [0,1]\}$ is compact and hence separable. Let Y be the closed linear span of $\{F(t) : t \in [0,1]\}$. Then Y is separable and Y contains the set $\{f(t) : F'_{ap}(t) = f(t)\}$. Hence, the function f is essentially separably valued. It follows from the Pettis Measurability Theorem that f is measurable.

COROLLARY 4. *Let f be McShane integrable. If X is separable, then f is measurable.*

PROOF. By the proof of Theorem 3, f is weakly measurable. Since X is separable, f is separably valued. Hence f is measurable.

The next theorem was proved by R.A.Gordon in [7].

THEOREM 5. *If* $f : [0,1] \longrightarrow X$ *is Bochner integrable on* $[0,1]$, *then* f *is McShane integrable on* $[0,1]$.

PROOF. Since f is measurable there exist $E \subset [0,1]$ with $\mu(E) = 1$ and a sequence $\{f_n\}$ of countably-valued functions such that for each n the inequality $||f_n(t) - f\chi_E(t)|| \leq \frac{1}{n}$ holds for all t in $[0,1]$. It is clear that each f_n is Bochner integrable on $[0,1]$.

For each n let

$$f_n = \sum_{k=1}^{\infty} x_k^n \chi_{E_k^n}$$

where the sets $\{E_k^n : k \geq 1\}$ are disjoint and measurable. The series $\sum_k \mu(E_k^n) x_k^n$ is absolutely convergent and hence unconditionally convergent for each n. By([7], Theorem 15) each of the functions f_n is McShane integrable on $[0,1]$. Since $f\chi_E$ is the uniform limit of $\{f_n\}$ on $[0,1]$, the function $f\chi_E$ is McShane integrable on $[0,1]$ by ([7], Theorem 12). By ([7], Theorem 6), the function f is McShane integrable on $[0,1]$.

From Example 2, we note that a McShane integrable function need not be Bochner integrable.

THEOREM 6. *Let* $f : [0,1] \longrightarrow X$ *be McShane integrable on* $[0,1]$ *and let* $F(t) = \int_a^t f$. *If* F *is approximately differentiable* a.e. *on* $[0,1]$, $F'_{ap} = f$ a.e. *and* F'_{ap} *is bounded* a.e. *on* $[0,1]$, *then* f *is Bochner integrable on* $[0,1]$.

PROOF. If F is approximately differentiable a.e. on $[0,1]$ and $F'_{ap} = f$ a.e., then f is measurable by Theorem 3. Since F'_{ap} is bounded a.e. on $[0,1]$, $||f||$ is bounded a.e. and measurable. Hence f is Bochner integrable on $[0,1]$.

From the definitions, we note that a McShane integrable function f is Henstock integrable. It is well-known that a McShane integrable function is Pettis integrable ([5], 2C Theorem). A measurable Pettis integarable function is McShane integrable ([7], Theorem 17).

A vector-valued function is McShane integrable if and only if it is both Henstock integrable and Pettis integrable ([4], Theorem 8).

THEOREM 7. *Suppose that X contains no copy of c_0 and let $f : [0,1] \longrightarrow X$ be Dunford integrable on $[0,1]$. If f is Henstock integrable on $[0,1]$, then f is Pettis integrable on $[0,1]$.*

PROOF. If f is Henstock integrable on [0,1], then $f\chi_I$ is Henstock integrable for every closed subinterval of $[0,1]$ ([4], Proposition 4).

Since f is Dunford integrable, for every closed subinterval I of $[0,1]$ and every $x^* \in X^*$ we have

$$x^*[(H) - \int_I f] = (H) - \int_I x^* f = (L) - \int_I x^* f = [(D) - \int_I f](x^*)$$

From this equality, we have $(D) - \int_I f = (H) - \int_I f \in X$ for every closed subinterval I of $[0,1]$.

Since the Dunford integral of f on sets of Lebesgue measure zero is zero, $(D) - \int_I f \in X$ for every subinterval I of $[0,1]$. By ([7], Theorem 18), f is Pettis integrable on $[0,1]$.

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Department of Mathematics
Chungnam National University
Taejon 305-764, Korea
and
*
Department of Mathematics
Kongju National University
Kongju 314-701, Korea