JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 8, June 1995

A NOTE ON THE COMPLEXIFICATION OF CONFORMAL GROUP *II**

LEE, KE-SEUNG

ABSTRACT. In the white noise analysis the one-parameter groups play the powerful role. In this report, we will see a subgroup of infinite dimensional unitary group U_∞ including guage transform and structure of this subgroup under the view point of Lie algebra.

0. Introduction

There have been several ways to the analysis of complex Brownian functionals. One of the ways is the use of transform groups as seen in real white noise analysis.

In section 1 and section 2 we shall give basic definitions and properties of complex white noise space.

Section 3 will devoted to the study of unitary group coming from complexification of real conformal group.

In section 4 we will find the structure of the unitary group including guage transform.

As mentioned in concluding remarks the author hope that the results obtained in section 4 would successfully be applied to variational calculus in multi- dimensional generalized Brownian functionals.

1. Complex White noise

We start with complexification of white noise.Let

$$E \subset L^2(R^d) \subset E^* \tag{1}$$

Received by the editors on June 30, 1995.
1991 *Mathematics subject classifications*: Primary 60H30, 22E65.

be a Gelfand triple with white noise μ. We now form a complexification of E,

(2) $$E_c = E + iE$$

which is a collection of complex valued functions of the form

$$\zeta = \xi + i\eta, \qquad \xi, \eta \in E$$

Obviously E_c is a subspace of complex Hilbert space $L_c^2(R^d)$ and the inner product $(\ ,\ )_c$ is given by

(3) $$(\zeta_1, \zeta_2)_c = \{(\xi_1, \xi_2) + (\eta_1, \eta 2)\} + i\{(\eta_1, \xi_2) - (\xi_1, \eta_2)\}$$

with $\zeta_k = \xi_k + i\eta_k, k = 1, 2$.

The dual space can be taken as

(4) $$E_c^* = E^* + iE^*$$

where the pairing

(5)
$$< z, \zeta >_c = (< x, \zeta > + < y, \eta >) + i(- < x, eta > + < y, \xi >),$$
$$z \in E_c^*, \zeta \in E_c.$$

Hence the pairing $<, >_c$, which links E_c^* and E_c, is linear in z and is antilinear in ζ.

The σ-algebra $\mathcal{B}$ on E_c^* is the one generated by cylinder sets which make the pairing $< z, \zeta >_c$ being measurable with respect to z for any $\zeta \in E_c$.

Let μ_1, μ_2 be the measures of white noise with variance $1/2$ given on the space E^* and iE^*, respectively. We introduce the product measure $\mu_1 \times \mu_2 = \nu$ on E_c^*. Combining these, we obtain a measure space $(E_c^*, \mathcal{B}, \nu)$.

DEFINITION 1. The triple $(E_c^*, \mathcal{B}, \nu)$ is called a complex white noise space.

Properties on the complex white noise can be found in [Hida 71],[Hida 80].

2. Infinite dimensional unitary group

Let us introduce the collection U_∞ of all linear transforms g on E_c satisfying the following two conditions;

(i) g is homeomorphism of E_c
(ii) $\|g\zeta\| = \|\zeta\|, \zeta \in E_c(\| \quad \| \quad L_c^2(R^d) - \text{norm})$.

Obviously U_∞ becomes a groups under the usual product

$$(g_1 g_2)\zeta = g_1(g_2\zeta).$$

DEFINITION 2. The group U_∞ is called the infinite dimensional unitary group.

For any $g \in U_\infty$, we can define the canonical adjoint g^* of g by the canonical bilinear form;

$$< z, g\zeta >=< g^* z, \zeta >, \qquad z \in E_c^*, \zeta \in E_c.$$

The operator g^* is a linear isomorphism of E_c^*. The collection $g^*, g \in E_c$, again forms a group and denoted it by U_∞^*.

PROPOSITION 1. *For any $g \in U_\infty$, it holds that*

$$g^* \circ \nu = \nu. \tag{6}$$

COROLLARY. *The operator U_g given by*

$$U_g\phi(z) = \phi(g^*z) \tag{7}$$

is a unitary operator acting on $(L_c^2) = L^2(E_c^*, \nu)$ and the group $U_\infty^* = \{U_g : g \in U_\infty\}$ is isomorphic to U_∞.

The proofs of Proposition 1 and Corollary can be found in Hida[80].

3. Guage transform

We are interested in conformal group related to the guage transform. We omit the detail on real conformal group, which can be found in [HLL85].

Let ψ_t be a diffeomorphism on R^d with parameter $t \in R^1$. Then the transform g_t below is linear homeomorphism on E_c and $L_c^2(R^d)$-norm preserving:

$$(g_t\zeta)(u) = \exp(ih(t,u))\zeta(\psi_t(u))\sqrt{|\psi_t'(u)|} \tag{8}$$

where $h(t,u)$ is real valued function and $|\psi_t'(u)|$ is Jacobian.

We want the collection $\{g_t : t \in R\}$ to be a one parameter group. The one parameter group property

$$g_t g_s = g_{t+s}$$

requires the following functional equations;

$$h(s,u) + h(t, \psi_s(u)) = h(t+s, u) \tag{9}$$

$$\psi_t(u) = K^{-1}(K(u) + t \cdot C) \tag{10}$$

where K is bijective map on R^d and C is a d-dimensional vector. See [HLL85] and [Hida 88].

THEOREM 2. *If $\{g_t\}$ satisfies the one parameter group property then the infinitesimal generator of $\{g_t\}$ has the form below under the suitable condition on $h(t,u)$;*

$$(a,\nabla)+\frac{1}{2}(\nabla,a)+i(\frac{\partial h}{\partial t}|_{t=0})I, \tag{11}$$

where $i=\sqrt{-1}, a=C\cdot^t(\frac{\partial K^{-1}}{\partial v})_{K(u)}$ and $\nabla=(\frac{\partial}{\partial x_1},\cdots\frac{\partial}{\partial x_d})$.

The Proof of Theorem 2 is similar to the one in [HLL85], hence we omit the proof.

DEFINITION 3. The one-parameter group $\{I_t\}$,

$$(I_t\zeta)(u)=exp(it)\zeta(u) \tag{12}$$

is called guage transform.

Even though, guage transform is the simplest one-parameter group, it will play special role in our analysis of complex white noise functionals.

Clearly, the infinitesimal generator of $\{I_t\}$ is iI.

Here is the appropriate place for us to introduce the complex conformal group in E_c^*. In what follows the basic nuclear space E is taken to the one defined by

$$D_0(R^d)=\{\xi:\xi \text{ and } \omega\xi \text{ are } C^\infty - \text{ function on } R^d\}$$

where ω denote the inversion

$$(\omega\xi)(u)=\xi(\frac{u}{|u|^2})|u|^{-d}, \qquad u\in R^d.$$

It readily follows that the space $D_0(R^d)$ turns out to suitable for our purpose.

DEFINITION 4. The complex conformal group on E_c^* is the collection of the adjoints of the following transforms on E_c where $\zeta \in E_c$:

(i) shifts $\{S_t^j | t \in R, j = 1, \cdots, d\}$

$$(S_t^j)(u) = \zeta(u - te_j) \tag{14}$$

where e_j is standard unit vector on R^d,

(ii) isomorphic dilation $\{D_t | t \in R\}$

$$(D_t\zeta)(u) = \zeta(e^t u)e^{\frac{td}{2}}, \tag{15}$$

(iii) rotation $\{R_\theta^{kj} | k \neq j, \quad j = 1, \cdots, d, \theta \in R\}$

$$(R_\theta\zeta)(u) = \zeta(r_\theta^{kj} u), \tag{16}$$

where r_θ^{kj} is Euler angle in the $k - j$ plane,

(iv) special conformal transform $\{K_t^j | t \in R \quad j = 1, \cdots, d\}$

(17)

$$(K_t^j\zeta)(u) = ((\omega S_t^j \omega\zeta)(u) = \zeta(\frac{u - t|u|^2 e_j}{1 - 2tu_j + t^2|u|^2})|1 - 2tu_j + t^2|u|^2|^{\frac{-d}{2}}$$

REMARK. The complex conformal group generated by the above four transforms (i),(ii),(iii),(iv) is isomorphic to real conformal group. See[HLL 85].

4. Structures of unitary subgroup of U_∞^*

In this section, we discuss the structure of complex conformal group in section 3 related to guage trandforms.

If $h(t, u)$ has the form

$$h(t, u) = f(\psi_t(u)) - f(u) \tag{18}$$

where f is any real valued function, then $h(t,u)$ satisfy the equation(9). If we assume that the function h in (8) has the form in (18) and the function f in (18) is continuously differentiable, then we get the explicit infinitesimal generators in Theorem 2 for complex conformal group on E_c^* as follow:

(i′) for shifts

$$S^j = -\frac{\partial}{\partial u_j} - i\frac{\partial h_1}{\partial u_j}I, \tag{13′}$$

(ii′) for isomorphic diliation

$$D = \Sigma u_j\frac{\partial}{\partial u_j} + \frac{d}{2} + i(\Sigma u_j\frac{\partial h_2}{\partial u_j})I, \tag{14′}$$

(iii′) for rotation R^{kj}

$$R^{kj} = u_k\frac{\partial}{\partial u_j} - u_j\frac{\partial}{\partial u_k} + i(u_j\frac{\partial h_3}{\partial u_k} - u_i\frac{\partial h_3}{\partial u_j})I, \tag{15′}$$

(iv′) for special conformal transform SC^j

$$SC^j = 2u_j(u,\nabla) - |u|^2\frac{\partial}{\partial u_j} + du_j + i(\frac{\partial h_4}{\partial u_j}(-|u|^2 + 2u_j^2)I, \tag{16′}$$

where $i = \sqrt{-1}$ and $j,k = 1,\cdots,d, j \neq k$.

We now come to our main theorem.

THEOREM 3. *If we take the function h_j being $h(u) = u_1 + \cdots + u_d$ in the above formulas $(i'),(ii'),(iii'),(iv')$, then the algebra generated by*

$$\{iI, S^j, D, R^{kj}, SC^j | k \neq j, k,j = 1,\cdot,d\}$$

forms a $(\frac{d(d+3)}{2}+2)$*-dimensional semi-simple Lie algebra.*

PROOF. For simplicity we will fix $d=2$. The commutation relations listed below can be easily computed. Set $D'=D-I$, where I is identity operator. Then we have

$$[S^1,S^2]=0, \quad [S^1,D']=S^1, \quad [S^2,D']=S^2$$
$$[S^1,R]=S^2, \quad [S^2,R]=-S^1$$
$$[S^1,SC^1]=-2D'-2I, \quad [S^1,SC^2]=2R$$
$$[S^2,SC^1]=-2R, \quad [S^2,SC^2]=2D'+2I$$
$$[D',R]=0, \quad [D',SC^1]=SC^1, \quad [D',SC^2]=SC^2$$
$$[R,SC^1]=-SC^2, \quad [R,SC^2]=SC^1, \quad [SC^1,SC^2]=0$$

This relations complete the proof.

CONCLUDING REMARKS

Before closing this article remarks are now in order. (1) It is clear that iI plays special role in the analysis of complex white noise functionals. (2) It will be interesting to find the connection between the complex conformal group and variational calculus in generalized Brownian functionals.

ACKNOWLEDGEMENTS

The author is glad to express his thanks tp Professor T.Hida and N.Obata. Professor Hida has encouraged the author to complete this article and professor Obata gave the author valuable comments on the Theorem 3.

REFERENCES

1. J. R. Aitchison[82], *An Informal Introduction to guage Field Theories*, Univ. Press.

2. T. Hida, I. Kubo, H. Nomoto and Yoshizawa[68], *On Projective Invariance of Brownian Motian*, Publ. PIMS,ser A.
3. T. Hida[71], *Complex White Noise and Infinite Dimensional Unitary Group*, Lecture note **3** (1971), Nagoya Univ.
4. T. Hida[79], *A Role of Fourier Transform in Theory of Infinite Dimensional Unitary Group*, J. Math (1973), 14-16, Kyota Univ.
5. T. Hida[80], *Brownian Motion*, Springer-Verlag.
6. T. Hida, H. H. Kuo, J. Pottoff and L. Striet[93], *White Noise:An Infinite Dimensional Calculus*, Kluwer Academy.
7. T. Hida, K. S. Lee and S. S. Lee[85], *Conformal Invariance of White Noise*, Nagoya Math. J **98** (1985), 87-98.
8. H. H. Kuo[92], *Lectures on White Noise Analysis*, Soochow J. of Math **18(3)** (1992), 229-302.
9. H. H. Kuo[93], *Analysis of White Noise Functionals*, Lecture Note (1993), Preprint.
10. K. S. Lee[90], *White Noise Approach to Gaussian Random Field*, Nagoya Math. J **119** (1990), 93-106.
11. H. Obata[92], *Rotation-Invariant Operators on White Noise Functionals*, Math. Z **210** (1992), 69-89.
12. N. Obata[94], *Elements of White Noise Calculus*, LNM(1537) (1994), Springer-Verlag.
13. H. Yoshizawa[70], *Rotation Group of Hilbert Space and Its Application to Brownian Motion*, in Proc.International Conference on Functional Analysis and Related Topics (1970), 414-423, Univ. of Tokyo Press.

DEPARTMENT OF MATHEMATICS
KOREA UNIVERSITY
CHOCHIWON 339-800,KOREA