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ON THE EXISENCE OF A HOLOMORPHIC STRUCTURE ON A SMOOTH COMPLEX VECTOR BUNDLE

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ABSTRACT. We give another detailed proof of the existence of a holomorphic structure on a smooth complex vector bundle over a complex manifold.

1. Introduction

A holomorphic vector bundle E over a complex manifold M is a complex vector bundle together with a complex manifold structure on E, such that for any $x \in M$, there exists $U \ni x$ in M and a trivialization

$$\phi_\alpha : E_{U_\alpha} \to U_\alpha \times \mathbb{C}^k$$

that is a biholomorphic map of complex manifolds. Such a trivialization is called a holomorphic trivialization. Note that if $\{\phi_\alpha : E_{U_\alpha} \to U_\alpha \times \mathbb{C}^k\}$ are holomorphic trivializations, then the transition functions for E relative to $\{\phi_\alpha\}$ are holomorphic maps, and that, conversely, given holomorphic maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{C})$ satisfying the identities

$$g_{\alpha\beta}(x) \cdot g_{\beta\alpha}(x) = I, \ \ x \in U_\alpha \cap U_\beta$$

$$g_{\alpha\beta}(x) \cdot g_{\beta\gamma}(x) \cdot g_{\gamma\alpha}(x) = I, \ \ x \in U_\alpha \cap U_\beta \cap U_\gamma,$$

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we can construct a holomorphic vector bundle $E \to M$ with transition functions $g_{\alpha\beta}$ ([GH]).

A condition for the existence of a holomorphic structure on a smooth complex vector bundle over a complex manifold is well-known as the following ([K]).

THEOREM 1. *Let E be a C^∞ vector bundle over a complex manifold M. Let $D = D' + D''$ be a connection on E. Then there is a unique holomorphic vector bundle structure on E such that $D = d''$ if $D'' \circ D'' = 0$.*

In the above theorem, D' and D'' are the $(1,0)$ and $(0,1)$ components of D, respectively.

We give another detailed proof of the above theorem by using the following.

THEOREM 2. *Let $\Gamma^i_{\lambda j}, 1 \le i,j \le r$, $1 \le \lambda \le n$, be C^∞ complex valued functions on $\mathbb{C}^n$ which satisfy the equations*

$$\frac{\partial \Gamma^i_{\lambda j}}{\partial \bar{z}^\mu} + \sum_k \Gamma^i_{\mu k}\Gamma^k_{\lambda j} = \frac{\partial \Gamma^i_{\mu j}}{\partial \bar{z}^\lambda} + \sum_k \Gamma^i_{\lambda k}\Gamma^k_{\mu j}, \ 1 \le i,j,k \le r, \ 1 \le \lambda, \mu \le n.$$

Then there exist r C^∞ functions $a^1, a^2, \cdots, a^r$ defined in a neighborhood of the origin in $\mathbb{C}^n$ which satisfy the equations

$$\frac{\partial a^i}{\partial \bar{z}^\mu} + \sum_j \Gamma^i_{\mu j} a^j = 0$$

with initial condition $a^i(0) = c_i$, $(c_1, c_2, \cdots, c_r) \in \mathbb{C}^r$.

2. Proofs of the Theorems

In the first we prove theorem 2. We need the following lemma.

LEMMA. *([N], [NN]) Let* $\Omega = \Omega(y)$ *be a* k*-dimensional subspace of complex valued forms of a* $2k$*-dimensional smooth manifold* M*, defined in a neighborhood of the origin and* $\Omega \cap \overline{\Omega} = 0$. *Then necessary and sufficient condition for the existence of a new local coordinate* $z = (z^1, z^2, \cdots, z^k) : M \to \mathbb{C}^k$ *such that* Ω *is spanned by* $< dz^a >, 1 \leq a \leq k$*, is* $d\Omega \subset$ *ideal generated by* Ω.

Moreover, the new coordinates in the above lemma have the following property: if, on a neighborhood, a basis for Ω can be chosen so that a finite number of forms $dy^j + \sqrt{-1}dy^{j+k}$ are basis elements, then for these values of j, we have $dz^j = dy^j + \sqrt{-1}dy^{j+k}$ ([N]).

PROOF OF THEOREM 2. Let $(z^1, z^2, \cdots, z^n)$ and $(\tilde{z}^1, \tilde{z}^2, \cdots, \tilde{z}^n, \zeta^1, \zeta^2, \cdots, \zeta^r)$ be standard coordinate functions on $\mathbb{C}^n$ and $\mathbb{C}^n \times \mathbb{C}^r$, respectively. Consider a space of forms Ω on $\mathbb{C}^n \times \mathbb{C}^r$ spanned by

$$\Omega = < d\tilde{z}^\lambda, d\zeta^i + \sum_j \omega^i_j \zeta^j >, \;\; 1 \leq i, j \leq r, 1 \leq \lambda \leq n,$$

where $\omega^i_j = \sum_\lambda \Gamma^i_{\lambda j} d\overline{\tilde{z}}^\lambda$. Then $\Omega \cap \overline{\Omega} = 0$ and $d\Omega \subset$ ideal generated by Ω. So by the lemma, there is a new coordinate $(w^1, w^2, \cdots, w^n, \eta^1, \eta^2, \cdots, \eta^r)$ on $\mathbb{C}^n \times \mathbb{C}^r$ such that Ω is spanned by $< dw^\lambda, d\eta^i >, 1 \leq \lambda \leq n, 1 \leq i \leq r$, and we have $\tilde{z}^\lambda = w^\lambda, 1 \leq \lambda \leq n$.

Now consider ζ^i as a function of $w^1, w^2, \cdots, w^n, \eta^1, \eta^2, \cdots, \eta^r$. Then we have

$$\begin{aligned}
0 &\equiv d\zeta^i + \sum_j \omega^i_j \zeta^j \;\; (\mathrm{mod}\ \Omega) \\
&\equiv \sum_j \frac{\partial \zeta^i}{\partial \tilde{\eta}^j} d\tilde{\eta}^j + \sum_\lambda \frac{\partial \zeta^i}{\partial \tilde{w}^\lambda} d\tilde{w}^\lambda + \sum_j (\sum_\lambda \Gamma^i_{\lambda j} d\overline{\tilde{z}}^\lambda) \;\; (\mathrm{mod}\ \Omega) \\
&= \sum_j \frac{\partial \zeta^i}{\partial \overline{\eta}^j} d\overline{\eta}^j + \sum_\lambda (\frac{\partial \zeta^i}{\partial \overline{w}^\lambda} + \sum_j \Gamma^i_{\lambda j} \zeta^j) d\overline{w}^\lambda.
\end{aligned}$$

So we have

$$\frac{\partial \zeta^i}{\partial \overline{\eta}^j} = 0, \quad \frac{\partial \zeta^i}{\partial \overline{w}^\lambda} + \sum_j \Gamma^i_{\lambda j}\zeta^j = 0.$$

Now define functions $a^i, 1 \le i \le r$, on $\mathbb{C}^n$ by

$$a^i(z) = \zeta^i(\phi^{-1}(z,0)),$$

where $\phi : \mathbb{C}^n \times \mathbb{C}^r = (\tilde{z}^1, \tilde{z}^2, \cdots, \tilde{z}^n, \zeta^1, \zeta^2, \cdots, \zeta^r) \to \mathbb{C}^n \times \mathbb{C}^r = (w^1, w^2, \cdots, w^n, \zeta^1, \zeta^2, \cdots, \zeta^r)$ is a local diffeomorphism such that $\phi(0,c) = (0,0)$. Here $c = (c_1, c_2, \cdots, c_r) \in \mathbb{C}^r$ and $(a^1(0), a^2(0), \cdots, a^r(0)) = (\zeta^1(\phi^{-1}(0,0)), \zeta^2(\phi^{-1}(0,0)), \cdots, \zeta^r(\phi^{-1}(0,0)) = (c_1, c_2, \cdots, c_r)$.

Then

$$\begin{aligned} \frac{\partial a^i(z)}{\partial \overline{z}^\mu} + \sum_\mu \Gamma^i_{\mu j} a^\mu(z) &= \frac{\partial \zeta^i(\phi^{-1}(z,0))}{\partial \overline{w}^\mu} + \sum_\mu \Gamma^i_{\lambda j}\zeta^j(\phi^{-1}(z,0)) \\ &= 0. \end{aligned}$$

Thus $a^i(z), 1 \le i \le r$, are the solutions.

PROOF OF THEOREM 1. Let $\tilde{s}_1, \tilde{s}_2, \cdots, \tilde{s}_r$ be linearly independent C^∞ local sections for E on U and let $D''\tilde{s}_j = \sum_i \omega^i_j \tilde{s}_i$, where ω^i_j is the $(0,1)$ part of the connection form of D with respect to $\tilde{s}_1, \tilde{s}_2, \cdots, \tilde{s}_r$.

Now let $s_j = \sum_i a^i_j \tilde{s}_i$. Then

$$\begin{aligned} D'' s_j &= D''(\sum_i a^i_j \tilde{s}_i) \\ &= \sum_i (\overline{\partial} a^i_j \cdot \tilde{s}_i + a^i_j D''\tilde{s}_i) \\ &= \sum_i \overline{\partial} a^i_j \cdot \tilde{s}_i + \sum_{i,k} a^i_j \omega^k_i \tilde{s}_k \\ &= \sum_k (\overline{\partial} a^k_j + a^i_j \omega^k_i)\tilde{s}_k. \end{aligned}$$

So $\{s_1, s_2, \cdots, s_r\}$ is a holomorphic local frame on U if $\overline{\partial} a_j^k + \sum_i \omega_i^k a_j^i = 0$ for some C^∞ function (a_j^i) with $\det(a_j^i) \neq 0$. This equation has a solution by the theorem 2.

Now let U_α, U_β be two overlapping open sets of M and let $\{s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_r}\}$ and $\{s_{\beta_1}, s_{\beta_2}, \cdots, s_{\beta_r}\}$ be two holomorphic local frames for E on U_α, U_β, respectively, such that $D'' s_{\alpha_i} = 0, D'' s_{\beta_j} = 0$. Then on $U_\alpha \cap U_\beta, s_{\alpha_j} = \sum_i g_j^i s_{\beta_i}$, where $g = (g_j^i)$ is a transition map. Then

$$\begin{aligned} 0 &= D'' s_{\beta_j} \\ &= D''(\sum_i g_j^i s_{\alpha_i}) \\ &= \sum_i (\overline{\partial} g_j^i s_{\alpha_i} + g_j^i D''(s_{\alpha_i})) \\ &= \sum_i (\overline{\partial} g_j^i s_{\alpha_i}). \end{aligned}$$

So $\overline{\partial} g_j^i = 0$. Thus the transition map $g = (g_j^i)$ is holomorphic.

So there is a holomorphic structure on E and the uniqueness is trivial.

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