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ANALYTIC OPERATOR–VALUED FEYNMAN INTEGRAL ASSOCIATED WITH REGULAR DIRICHLET FORM

KI SEONG CHOI

ABSTRACT. I study the extension of the definition of the analytic operator-valued Feynman integral to potentials given by a class of signed measures described in terms of additive functionals associated with regular Dirichlet forms.

0. Introduction

The present state of the existence theory for the analytic operator-valued Feynman intrgral extends considerably further [13] than seems to be generally known.

In their recent papers[2], they show that the definition of analytic operator- valued Feynman integral is extended to potentials given by a class of signed measures described in terms of additive functionals associated with Dirichlet forms.

But their results are related with particular cases of smooth measures for the classical Dirichlet form associated with the Lalpacian. It is natural ask ourselves what happens if one tries to carry through similar constructions using an arbitrary smooth measures associated with a general(regular) Dirichlet form. In this paper, we initiate such a study.

We consider a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X, m)$ where X is a locally compact separable metric space and m is a positive Radon

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measure on X with $\text{supp}[m] = X$. Let $M = (\Omega, X_t, \zeta, P_x)$ be a Hunt process on X which is m-symmetric and associated with $(\mathcal{E}, \mathcal{F})$.

A function $A : [0,\infty) \times \Omega \to [-\infty, \infty]$ is said to be an AF (*additive functional*) if

(1) $A_t(\cdot)$is F_t-measurable, where F_t is the smallest completed σ-algebra which contains $\sigma\{X_s : s \leq t\}$.;

(2) there exist a defining set a $\Lambda \in F_\infty$ and an exceptional set $N \subset X$ with $\text{Cap}(N) = 0$ such that $P_x(\Lambda) = 1$ for all $x \in X - N, \theta_t\Lambda \subset \Lambda$ for all $t > 0$ (θ_t denotes the shift operator on Ω) and for each $\omega \in \Lambda, A_0(\omega) = 0, |A_t(\omega)| < \infty$ for $t < \zeta(\omega), A.(\omega)$ is right contiuous and has left limit, $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t\omega)$ for $s, t \geq 0$.

An additive functional A is called a PCAF (*positive continuous* AF) if A is an additive functional and $A.(\omega)$ is non-negative and continuous function for each ω in its defining set Λ.

Given a PCAF A, there exists a unique Borel measure μ on X, which is called the *Revuz measure of* A such that

$$\lim_{t\downarrow 0} \frac{1}{t} E_{h\cdot m}[\int_0^t (f(X_s)\, dA_s] = \langle f \cdot \mu, h\rangle := \int_X h(x)(f\cdot\mu)(dx)$$

for all γ-excessive functions h and $f \in \mathcal{B}^+$ ($\mathcal{B}^+$ denotes all non-negative Borel functions on $X, \gamma \geq 0$ is a constant).

Denote by S the totality of the associated Revuz measures of PCAF's. The elements in S are called smooth measures. A simple analytical description of S has been given as follows [10].

For a signed Borel measure $\mu = \mu^+ - \mu^-$, we write $\mu \in S - S$ (resp, $S - S_{K_0}$, etc), if $\mu^+ \in S$ (resp, S_{K_0}, etc), and $\mu^- \in S$ (resp, $S - S_{K_0}$, etc).

For a given smooth measure μ, we denote by A^μ the unique (up to equivalence class) positive continuous additive functional such that μ

is the Revuz measure of A^{μ}. In this case, if $\mu \in S - S$, we denote $A^{\mu^+} - A^{\mu^-}$ by A^{μ}, and μ is still referred to as the Revuz measure of A^{μ}.

In this paper, l will find some condition for the existenc of the analytic operator-valued Feynman integral associated with regular Dirichlet form. This paper is organized as follows. In section 1, l will discuss Feynman-Kac formula associated with regular Dirichlet form in the complex setting. In section 2, l will prove some existence theorem for the analytic operator-valued Feynman integral related with some smooth measures μ.

1. Feynman-Kac formula related with regular Dirichlet form

Throughout this paper, $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(X, m)$, where X is a locally compact separable metric space. S_0 is the subfamily of S which consists of all Radon measures of finite energy integral.

For $\alpha \geq 0, \mu$ and v in $S - S, f \in \mathcal{B}(X)$, we set

$$U_v^{\alpha+\mu} f(x) = E_x[\int_0^\infty e^{-\alpha t - A_t^\mu} f(X_t)\, dA_t^v]$$

provided the right-hand side makes sense. When $v = m$, we simply write $U^{\alpha+\mu} f$ for $U_v^{\alpha+\mu} f$. we set

$$\mu T_t f = E_\mu[\int_0^t f(X_s)\, ds]$$

$$\mu U^\alpha f = E_\mu[\int_0^\infty e^{-\alpha s} f(X_s)\, ds]$$

provided the intrgrals make sense.

THEOREM 1. *Let μ be a smooth measure. Then the following assertions are equivalent to each other.*

(1) *μU^α is a bounded functional on $L^1(X;m)$ for each $\alpha > 0$ and $\lim_{\alpha\uparrow\infty} \|\mu U^\alpha\| = 0.$;*
(2) *μT_t is a bounded functional on $L^1(X;m)$. for each $t > 0$ and $\lim_{t\downarrow 0} \|\mu T_t\| = 0;$*
(3) *$\lim_{\alpha\uparrow\infty} \|U_\mu^\alpha 1\|_q = 0.$;*
(4) *$\lim_{t\downarrow 0} \|EA_t^\mu\|_q = 0.$;*
In (1) and (2) $\|\cdot\|$ denote the operator norm of a functional on $L^1(X,m)$.

PROOF. See[4]

DEFINITION 1. A smooth measure μ is said to belong to the Kato class and is denoted $\mu \in S_K$ if any one of the assertions of Theorem 1 is true. Let us define the family S_{K_0} as follows

$$S_{K_0} = \{\mu \in S_K \cap S_0 : \mu(X) < \infty\}.$$

From now on, let us use the short notation $L^2(\mu)$ for $L^2(X,\mu)$. For $\mu \in S$, we set

$$Q_\mu(f,g) = \int_X f(x)\cdot g(x)\mu(dx), \quad \forall f,g \in L^2(|\mu|+m)$$

one can show that $L^2(|\mu|+m)$ is dense in $L^2(m)$. Hence Q_μ is a quadratic form on $L^2(m)$. We put

$$\mathcal{E}_\mu(f,g) = \mathcal{E}(f,g) + Q_\mu(f,g), \quad \forall f,g \in \mathcal{F}^\mu$$

where

$$\mathcal{F}^\mu = \mathcal{F} \cap L^2(|\mu|+m)$$

Let us introduce the notation

$$P_t^\mu f(x) = E_x[e^{-A_t^\mu} f(X_t)]$$

provided the right hand side makes sense. Notice that $(P_t^\mu)_{t\geq 0}$ is the so called Feynman-Kac functional.

THEOREM 2. *Let $\mu \in S - S_{k_0}$. Then*

(1) *$(\mathcal{E}_\mu, \mathcal{F}^\mu)$ is a lower semibounded closed quadratic form.*
(2) *$(\mathcal{F}^\mu) = \mathcal{F} \cap L^2(\mu^+ + m)$.*
(3) *$(P_t^\mu)_{t\geq 0}$ is the unique strongly continuous symmetric semigroup corresponding to $(\mathcal{E}_\mu, \mathcal{F}^\mu)$. Moreover, let $-H^\mu$ be the generator of (P_t^μ); then*

$$(H^\mu f, g) = \mathcal{E}^\mu(f, g)$$

PROOF. See[4] Proposition 3.1.

THEOREM 3. *Let $\mu \in S - S_{K_0}$; then*

$$(e^{-tH^\mu} u)(x) = P_t^\mu u(x) = E_x\{e^{-A_t^\mu(\omega)} u(\omega(t))\}$$

for every $u \in L^2(\mathbb{R}^d)$, and $m - a.e. x \in \mathbb{R}^d$. Here $-H^\mu$ is the generator of $(P_t^\mu)_{t\geq 0}$.

PROOF. Since $(e^{-tH^\mu})_{t\geq 0}$ and $(P_t^\mu)_{t\geq 0}$ are strongly coninuous semigroup possessing $-H^\mu$ as its generator, (P_t^μ) is equal to e^{-tH^μ}.

It is essential to quantum mechanics and the Feynman integral that the functions be from the space $L^2(\mathbb{R}^d, \mathbb{C})$ (over the scalar field $\mathbb{C}$) of square-integrable, complex valued functions. If ψ is a function in $L^2(\mathbb{R}^d, \mathbb{C})$ we denote by ψ_1, its real part and by ψ_2 its imaginary part; i.e. $\psi = \psi_1 + i\psi_2$.

THEOREM 4. *$\mathcal{E}_\mu$ is bounded below, densely defined and closed. We extend $\mathcal{E}_\mu$ to the subspace $D(\mathcal{E}_\mu^C) = D(\mathcal{E}_\mu) + iD(\mathcal{E}_\mu)$ of $L^2(\mathbb{R}^d, \mathbb{C})$ by*

$$\mathcal{E}_\mu^C(\psi, \varphi) = \mathcal{E}_\mu(\psi_1, \varphi_1) + \mathcal{E}_\mu(\psi_2, \varphi_2) + i[\mathcal{E}_\mu(\psi_2, \varphi_1) - \mathcal{E}_\mu(\psi_1, \varphi_2)].$$

Then $\mathcal{E}_\mu^C$ is a symmetric sesquilinear form on $D(\mathcal{E}_\mu^C)$ satisfying

(1) *$\mathcal{E}_\mu^C$ is bounded below by M where M is the lower bound for $\mathcal{E}_\mu$.*

(2) *$\mathcal{E}_\mu^C$ is densely defined.*

(3) *$\mathcal{E}_\mu^C$ is closed.*

PROOF. (1) Let M be such that $\mathcal{E}_\mu(u, u) \geq M\|u\|^2$ for all $u \in D(\mathcal{E})$. Let $\psi = \psi_1 + i\psi_2 \in D(\mathcal{E}_\mu) + iD(\mathcal{E}_\mu)$. Using the symmetry of $\mathcal{E}_\mu$, we get

$$\mathcal{E}_\mu^C(\psi, \psi) \geq M[\|\psi_1\|^2 + \|\psi_2\|^2] = M\|\psi\|^2$$

Symmetric follows form (1) [16,p276].

(2) Since $D(\mathcal{E}_\mu)$ is dense in $L^2(\mathbb{R}^d), D(\mathcal{E}_\mu) + iD(\mathcal{E}_\mu)$ is dense in $L^2(\mathbb{R}^d, \mathbb{C})$.

(3) Let $\{\psi_n\}$ be a sequence in $D(\mathcal{E}_\mu^C)$ such that $|||\psi_n - \psi_m||| \to 0$ as $n, m \to \infty$. Here $|||\psi|||^2 = \mathcal{E}_\mu^C(\psi, \psi) + (-M+1)\|\psi\|^2$. For each n, $\psi_n = \psi_{n,1} + i\psi_{n,2}$ where $\psi_{n,1}, \psi_{n,2}$ are in $D(\mathcal{E}_\mu)$.

$$\begin{aligned}
&|||\psi_n - \psi_m|||^2 \\
&= \mathcal{E}_\mu^C(\psi_n - \psi_m, \psi_n - \psi_m) + (-M+1)\|\psi_n - \psi_m\|^2 \\
&= \mathcal{E}_\mu(\psi_{n,1} - \psi_{m,1}, \psi_{n,1} - \psi_{m,1}) + \mathcal{E}_\mu(\psi_{n,2} - \psi_{m,2}) \\
&\quad + (-M+1)\|\psi_{n,1} - \psi_{m,1}\|^2 + (-M+1)\|\psi_{n,2} - \psi_{m,2}\|^2 \\
&= \mathcal{E}_\mu(\psi_{n,1} - \psi_{m,1}, \psi_{n,1} - \psi_{m,1}) + (-M+1)\|\psi_{n,1} - \psi_{m,1}\|^2 \\
&\quad + \mathcal{E}_\mu(\psi_{n,2} - \psi_{m,2}, \psi_{n,2} - \psi_{m,2}) + (-M+1)\|\psi_{n,2} - \psi_{m,2}\|^2 \\
&= |||\psi_{n,1} - \psi_{m,1}|||^2 + |||\psi_{n,2} - \psi_{m,2}|||^2.
\end{aligned}$$

The fact that $|||\psi_n - \psi_m||| \to 0$ implies that $|||\psi_{n,1} - \psi_{m,1}||| \to 0$ and $|||\psi_{n,2} - \psi_{m,2}||| \to 0$. Since $\mathcal{E}_\mu$ is closed, there exists $\psi_1, \psi_2 \in D(\mathcal{E}_\mu)$ such that $|||\psi_{n,1} - \psi_1||| \to 0$ and $|||\psi_{n,2} - \psi_2||| \to 0$ as $n \to \infty$. So $\mathcal{E}_\mu^C$ is closed.

THEOREM 5. *If $\mathcal{E}_\mu$ is bounded below, densely defined and closed, then there exists a unique, densely defined self-adjoint operator H^μ which is bounded below and satisfies $(H^\mu u, v) = \mathcal{E}_\mu(u, v)$ for all $u \in D(H^\mu)$ and $v \in D(\mathcal{E}_\mu)$*

PROOF. See[14] cor.24 and Theorem 2.6, p323.

Under the conditions of Theorem 4, there exists a unique densely defined (i.e., dense in $L^2(\mathbb{R}^d, \mathbb{C})$) self-adjoint operator H_C^μ and $\varphi \in D(\mathcal{E}_\mu^C)$.

Let L be a self-adjoint operator on a dense subspace $D(L)$ of $L^2(\mathbb{R}^d)$. If we define $L_\mathbb{C}$ on $D(L) + iD(L)$ by $L_\mathbb{C}(\psi_1 + i\psi_2) = L\psi_1 + iL\psi_2$, $L_\mathbb{C}$ is a self-adjoint operator on $D(L) + iD(L)$. For this reason, we get a self-adjoint operator $(H^\mu)_\mathbb{C}$ on $D(H^\mu) + iD(H^\mu) \subset D(\mathcal{E}_\mu) + iD(\mathcal{E}_\mu) = D(\mathcal{E}_\mu^\mathbb{C})$.

THEOREM 6. $H_\mathbb{C}^\mu = (H^\mu)_\mathbb{C}$.

PROOF. Using consequences [K, Cor. 2.4 and Th. 2.6,p323] of the First represtation Theorem and the simple fact (16,p279) that two self-adjoint operators, one of which extends the other, are actully equal, it suffices to show that for $\psi = \psi_1 + i\psi_2 \in D((H^\mu)_C)$ and $\varphi = \varphi_1 + i\varphi_2 \in D(\mathcal{E}_\mu^C)$, one has $((H^\mu)_C\psi, \varphi) = \mathcal{E}_\mu^C(\psi, \varphi)$. But this comes from the linearity of $(H^\mu)_C$

In prder to proceed to our result on the Feynman integral, we need to discuss the Feynman-Kac formula in the complex setting.

THEOREM 7. *Let L be a self-adjoint operator on a dense subspace $D(L)$ of $L^2(\mathbb{R}^d)$ with L bounded below by M. Then the linear extension $L_{\mathbb{C}}$ is the infinitesimal generator of a strongly continuous self-adjoint semigroup $(e^{-tL_{\mathbb{C}}}, t \geq 0)$ in $L^2(\mathbb{R}^d, \mathbb{C})$. For each $t \geq 0, e^{-tL_{\mathbb{C}}}$ is the liner extension of e^{-tL}.*

PROOF. See [14]

THEOREM 8. *Let $u \in S - S_{K_0}$ and H^μ is densely defined seif-adjoint operator which is bounded below and satisfies $(H^\mu u, v) = \mathcal{E}_\mu(u,v)$ for all $u \in D(H^\mu)$ and $v \in D(\mathcal{E}_\mu)$. Then the Feynman-Kac formula*

$$(1) \qquad (e^{-tH^\mu_{\mathbb{C}}}\psi)(x) = E_x\{e^{-A^\mu_t(\omega)}\psi(\omega(t))\}$$

holds for every $\psi \in L^2(\mathbb{R}^d, \mathbb{C}), m - a.e. x \in \mathbb{R}^d$, and all $t \geq 0$.

PROOF. This result follows immediately by Theorem 3, Theorem 7 and the linearity of the expectation E_X.

2. An existence of operator-valued function space integral related with some smooth measure μ

Now we are ready to give the definition of the analytic operator-valued Feynman integral for the special functions that concern us. Given $\omega \in \Omega = C([0,\infty), \mathbb{R}^d)$, let

$$(2) \qquad F^\mu(\omega) = e^{-A^\mu_t(\omega)}$$

DEFINITION 2 Given $t > 0, \psi \in L^2(\mathbb{R}^d, \mathbb{C})$ and $x \in \mathbb{R}^d$, consider the expression

$$(3) \qquad \begin{aligned}(J^t(F^\mu)\psi)(x) &= E_x\{e^{-A^\mu_t(\omega)}\psi(\omega(t))\} \\ &= \int_{\Omega_x} e^{-A^\mu_t(\omega)}\psi(\omega(t))\,dP_x(\omega)\end{aligned}$$

where Ω_x is the set of $\omega \in C([0,\infty),\mathbb{R}^d)$ such that $\omega(0) = x$ and P_x is the probability measure associated with the Brownian paths in $\mathbb{R}^d$ which start at x at time 0. The operator-valued function space integarl $J^t(F^\mu)$ exists for $t > 0$ if (3) defines $J^t(F^\mu)$ as an element of $\mathcal{L}(L^2(\mathbb{R}^d,\mathbb{C}))$, the space of bounded linear operators on $L^2(\mathbb{R}^d,\mathbb{C})$. If $J^t(F^\mu)$ exists for every $t > 0$ and, in addition, has an extension (necessarily unique) as a function of t to an analytic operator-valued function on $\mathbb{C}_+$ and a strongly continuous function on $\overline{\mathbb{C}_+}$, we say that $J^t(F^\mu)$ exists for all $t \in \overline{\mathbb{C}_t}$. When t is purely imaginary, $J^t(F^\mu)$ is called the analytic operator-valued Feynman integral of F.

THEOREM 10. *Let $\mu \in S - S_{K_0}$. The additive functional A_t^μ is related to the operator H^μ from*

$$(H^\mu u, v) = \mathcal{E}_\mu(u,v)$$

by the Feynman-Kac formula

$$(e^{-tH^\mu}u)(X) = E_x\{e^{-A_t^\mu(\omega)}u(\omega(t))\}.$$

Then the operator-valued function space integral $J^t(F^\mu)$ from Definition 2 exists for all $t \in \overline{\mathbb{C}_+}$ where F^μ is given by (2).

For $t \geq 0$, $J^t(F^\mu)$ is given for all $\psi \in L^2(\mathbb{R}^d,\mathbb{C})$ an m-a.e x by the Feynman-Kac formula (3). For all $t \in \overline{C}_+$,

$$J^t(F^\mu) = e^{-tH_{\mathbb{C}}^\mu}$$

where $e^{-tH_{\mathbb{C}}^\mu}$ is given meaning via the spectral theorem applied to the self-adjoint operator H_C^μ. In particalar, for $t \in R$, the analytic operator-valued Feynman integral $J^t(F^\mu)$ exists and we have

$$J^{it}(F^\mu) = e^{-itH_{\mathbb{C}}^\mu}$$

where $\{e^{-itH^{\mu}_{\mathbb{C}}} : t \in R\}$ is the unitary group (in $L^2(\mathbb{R}^d, \mathbb{C})$) corresponding to the self-adjoint operator $H^{\mu}_{\mathbb{C}}$.

PROOF. It suffices to show that the operator-valued function $A(t) = e^{-tH^{\mu}_{\mathbb{C}}}$ is defined and strongly continuous for all $t \in \overline{\mathbb{C}_+}$ and is analytic in $\mathbb{C}_+$.

For notational simplicity, we let $\mathcal{H} = H^{\mu}_{C}$. Since $\mathcal{H}$ is bounded below , $\sigma(\mathcal{H}) \subset [M, +\infty)$ for some $M > -\infty$. Here $\sigma(\mathcal{H})$ denotes the spectrum of $\mathcal{H}$. Since the function $g_t : \sigma(\mathcal{H}) \to \mathbb{C}$ defined by $g_t(u) = e^{-tu}$ is a bounded function of u for any $t \in \overline{\mathbb{C}_+}$, $g_t(\mathcal{H}) = A(t) = e^{-t\mathcal{H}}$ is defined and is a bounded linear operator on $L^2(\mathbb{R}^d, \mathbb{C})$ for all $t \in \overline{\mathbb{C}_+}$ (see, e.g.[16,pp259-264]).

Let $\{t_n\}$ be a sequence in $\overline{G}$ such that $t_n \to t$. Then $g_{t_n}(u) \to g_t(u)$ for all $u \in \omega(\mathcal{H})$. Further

$$\|g_{t_n}\|_{\infty} = sup\{g_{t_n}(u) \mid u \in \sigma(\mathcal{H})\}$$

is bounded. Also $g_{t_n}(\mathcal{H}) \to g_t(\mathcal{H})$ strongly. i.e., $A(t_n) = e^{-t_n\mathcal{H}} \to e^{-t\mathcal{H}} = A(t)$ in the strong operator topology as $n \to \infty$.

To show that $A(t) = e^{-t\mathcal{H}}$ is analytic in $\mathbb{C}_+$, it suffices to show that for every $\psi, \varphi \in L^2(\mathbb{R}^d, \mathbb{C})$, $(e^{-t\mathcal{H}}\psi, \varphi)$ is analytic in $\mathbb{C}_+$. We fix $\varphi \in L^2(\mathbb{R}^d, \mathbb{C})$ and we may as well even assume that $\|\varphi\| = 1$. Let P be the spectral measure associated with $\mathcal{H}$. The measure $\mu_{\varphi\varphi}$ defined for any Borel subset B of $\mathbb{R}$ by $\mu_{\varphi\varphi}(B) = (\varphi, P(B)\varphi)$ is a probability measure such that $\mu_{\varphi\varphi}(\sigma(\mathcal{H})) = 1$. Further, by the spectral theorem,

$$(e^{-t\mathcal{H}}\varphi, \varphi) = \int_{\mathbb{R}} e^{-tu} \, d\mu_{\varphi\varphi}(u) = \int_{M}^{\infty} e^{-tu} \, d\mu_{\varphi\varphi}(u)$$

Since $e^{-t\mathcal{H}}$ is strongly continuous in $\overline{\mathbb{C}_+}$, it certainly follows that

$(e^{-t\mathcal{H}}\varphi, \varphi)$ is continuous in $\mathbb{C}_+$. For any triangle Γ in $\mathbb{C}_+$,

$$\begin{aligned}\int_\Gamma (e^{-t\mathcal{H}}\varphi, \varphi)\, dt &= \int_\Gamma [\int_M^\infty e^{tu}\, du_{\varphi\varphi}(u)]\, dt \\ &= \int_M^\infty [\int_\Gamma e^{-tu}\, dt]\, d\mu_{\varphi\varphi}(u) \\ &= 0\end{aligned}$$

By the Morera's theorem, $(e^{-t\mathcal{H}}\varphi, \varphi)$ is analytic in $\mathbb{C}^+$. By a polarization argument, $(e^{-t\mathcal{H}}\psi, \varphi)$ is analytic in $\mathbb{C}_+$ for every $\psi, \varphi \in L^2(\mathbb{R}^d, \mathbb{C})$.

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Department of Mathematics
Keonyang University
Chungnam 320-800, Korea